YAU'S GRADIENT ESTIMATE AND LIOUVILLE THEOREM FOR POSITIVE PSEUDOHARMONIC FUNCTIONS IN A COMPLETE PSEUDOHERMITIAN MANIFOLD

*SHU-CHENG CHANG[1], *TING-JUNG KUO[2], AND JINGZHI TIE[3]

Abstract. In this paper, we first derive the sub-gradient estimate for positive pseudoharmonic functions in a complete pseudohermitian $(2n+1)$-manifold (M,J,θ) which satisfies the CR sub-Laplacian comparison property. It is served as the CR analogue of Yau's gradient estimate. Secondly, we obtain the Bishop-type sub-Laplacian comparison theorem in a class of complete noncompact pseudohermitian manifolds. Finally we have shown the natural analogue of Liouville-type theorems for the sub-Laplacian in a complete pseudohermitian manifold of vanishing pseudohermitian torsion tensors and nonnegative pseudohermitian Ricci curvature tensors.

1. Introduction

In [Y1] and [CY], S.-Y. Cheng and S.-T. Yau derived a well known gradient estimate for positive harmonic functions in a complete noncompact Riemannian manifold.

Proposition 1. ([Y1], [CY]) *Let M be a complete noncompact Riemannian m-manifold with Ricci curvature bounded from below by $-K$ $(K \geq 0)$. If $u(x)$ is a positive harmonic function on M, then there exists a positive constant $C = C(m)$ such that*

$$|\nabla f(x)|^2 \leq C(\sqrt{K} + \frac{1}{R}) \tag{1.1}$$

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on the ball $B(R)$ with $f(x) = \ln u(x)$. As a consequence, the Liouville theorem holds for complete noncompact Riemannian m-manifolds of nonnegative Ricci curvature.

In this paper, by modifying the arguments of [Y1], [CY] and [CKL], we derive a sub-gradient estimate for positive pseudoharmonic functions in a complete noncompact pseudohermitian $(2n+1)$-manifold (M, J, θ) which is served as the CR version of Yau's gradient estimate. Then we prove that the CR analogue of Liouville-type theorem holds for positive pseudoharmonic functions as well.

We first recall some notions as in section 2. Let (M, ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ, $\dim_{\mathbb{R}} \xi = 2n$. A CR structure J compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the integrability condition (see next section). A CR structure J can extend to $\mathbb{C}\otimes\xi$ and decomposes $\mathbb{C}\otimes\xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$ which are eigenspaces of J with respect to eigenvalues i and $-i$, respectively. A pseudohermitian structure compatible with ξ is a CR structure J compatible with ξ together with a choice of contact form θ and $\xi = \ker\theta$. Such a choice determines a unique real vector field T transverse to ξ which is called the characteristic vector field of θ, such that $\theta(T) = 1$ and $\mathcal{L}_T\theta = 0$ or $d\theta(T, \cdot) = 0$.

Let $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ be a frame of $TM\otimes\mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T_{0,1}$. We define Ric and Tor on $T_{1,0}$ by

$$Ric(X, Y) = R_{\alpha\bar{\beta}} X^\alpha Y^{\bar{\beta}} \tag{1.2}$$

and

$$Tor(X, Y) = i \textstyle\sum_{\alpha,\beta} (A_{\bar{\alpha}\bar{\beta}} X^{\bar{\alpha}} Y^{\bar{\beta}} - A_{\alpha\beta} X^\alpha Y^\beta). \tag{1.3}$$

Here $X = X^\alpha Z_\alpha$, $Y = Y^\beta Z_\beta$, ${R_\gamma}^\gamma{}_{\alpha\bar{\beta}}$ is the pseudohermitian curvature tensor, $R_{\alpha\bar{\beta}} = {R_\gamma}^\delta{}_{\alpha\bar{\beta}}$ is the pseudohermitian Ricci curvature tensor and $A_{\alpha\beta}$ is the torsion tensor.

In Yau's method for the proof of gradient estimates, one can estimate $\Delta(\eta(x)\,|\nabla f(x)|^2)$ for a nonegative cut-off function $\eta(x)$ on $B(2R)$ via Bochner formula and Laplacian comparison. At the end, one has gradient estimate (1.1) by applying the maximum principle to $\eta(x)\,|\nabla f(x)|^2$. However in order to derive the CR subgradient estimate, one of difficulties is to deal with the following CR Bochner formula (Lemma 9) which involving a term $\langle J\nabla_b\varphi,\ \nabla_b\varphi_0\rangle$ that has no analogue in the Riemannian case.

$$\begin{aligned}\Delta_b\,|\nabla_b\varphi|^2 &= 2\left|\left(\nabla^H\right)^2\varphi\right|^2+2\,\langle\nabla_b\varphi,\ \nabla_b\Delta_b\varphi\rangle\\ &+ (4Ric-2\,(n-2)\,Tor)\,((\nabla_b\varphi)_C\,,\ \ (\nabla_b\varphi)_C)+4\,\langle J\nabla_b\varphi,\ \nabla_b\varphi_0\rangle\,.\end{aligned}$$

Here $\left(\nabla^H\right)^2$, Δ_b, ∇_b are the subhessian, sub-Laplacian and sub-gradient respectively. We also denote $\varphi_0=T\varphi$.

In order to overcome this difficulty, we introduce a real-valued function $F(x,\ t,\ R,\ b)$: $M\times[0,\ 1]\times(0,\ \infty)\times(0,\ \infty)\to\mathbb{R}$ by adding an extra term $t\eta(x)\,f_0^2(x)$ to $|\nabla_b f(x)|^2$ as following

$$F(x,\ t,\ R,\ b)=t\left(|\nabla_b f(x)|^2+bt\eta(x)\,f_0^2(x)\right)$$

on the Carnot-Carathéodory ball $B(2R)$ with a constant b to be determined. In section 3, we derive the CR subgradient estimate (1.13) and (1.9) by applying the maximum principle to $\eta(x)\,F(x,t)$ for each fixed $t\in(0,1]$ if the CR sub-Laplacian comparison property (see Definition 2) holds on (M,J,θ).

Definition 1. Let (M,J,θ) be a pseudohermitian $(2n+1)$-manifold. A piecewise smooth curve $\gamma:[0,1]\to M$ is said to be horizontal if $\gamma'(t)\in\xi$ whenever $\gamma'(t)$ exists. The length of γ is then defined by

$$l(\gamma)=\int_0^1\langle\gamma'(t),\gamma'(t)\rangle_{L_\theta}^{\frac{1}{2}}dt.$$

Here $\langle\ ,\ \rangle_{L_\theta}$ is the Levi form as in (2.2). The Carnot-Carathéodory distance between two points $p,\ q \in M$ is

$$d_c(p,q) = \inf\{l(\gamma)\,|\,\gamma \in C_{p,q}\}$$

where $C_{p,q}$ is the set of all horizontal curves joining p and q. We say M is complete if it is complete as a metric space. We refer to [S] for some details. By Chow connectivity theorem [Cho], there always exists a horizontal curve joining p and q, so the distance is finite. Furthermore, there is a minimizing geodesic joining p and q so that its length is equal to the distance $d_c(p,q)$.

Definition 2. Let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold with

$$(2Ric - (n-2)Tor)\,(Z,\ Z) \geq -2k\,|Z|^2 \tag{1.4}$$

for all $Z \in T_{1,0}$, and k is an nonnegative constant. We say that $(M,\ J,\ \theta)$ satisfies the CR sub-Laplacian comparison property if there exists a positive constant $C_0 = C_0(k,n)$ such that

$$\Delta_b r \leq C_0(\frac{1}{r} + \sqrt{k}) \tag{1.5}$$

in the sense of distributions. Here Δ_b denote sub-Laplacian and $r\,(x)$ is the Carnot-Carathéodory distance from a fixed point $x_0 \in M$.

Let $(M,\ J,\ \theta)$ be the standard Heisenberg $(2n+1)$-manifold $(\mathbf{H}^n,\ \mathbf{J},\ \boldsymbol{\theta})$. We have $R_{\alpha\bar{\beta}} = 0$ and $A_{\alpha\beta} = 0$. Then the following CR sub-Laplacian comparison property holds on $(\mathbf{H}^n,\ \mathbf{J},\ \boldsymbol{\theta})$.

Proposition 2. ([CTW]) Let $(\mathbf{H}^n, J, \theta)$ be a standard Heisenberg $(2n+1)$-manifold. Then there exists a constant $C_1^{\mathbf{H}^n} > 0$

$$\Delta_b r^{\mathbf{H}^n} \leq \frac{C_1^{\mathbf{H}^n}}{r^{\mathbf{H}^n}}. \tag{1.6}$$

For completeness, we sketch the proof of [CTW] in appendix A.

In this paper, by applying the differential inequality for sub-Laplacian of Carnot-Caratheodory distance as in Lemma 17, we can generalize the following Bishop-type sub-Laplacian comparison property to a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing pseudohermitian torsion tensors (we refer to Theorem 19).

Theorem 3. Let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing pseudohermitian torsion tensors

$$A_{\alpha\beta} = 0$$

and

$$Ric\,(Z,\ Z) \geq -k\,|Z|^2$$

for all $Z \in T_{1,0}$ and k is an nonnegative constant. Then there exists a constant $C_0 > 0$

$$\Delta_b r \leq C_0(\frac{1}{r} + \sqrt{k}) \tag{1.7}$$

in the sense of distributions.

In order to have an analogue of Liouville-type theorem (see Corollary 6) for positive pseudoharmonic functions (i.e. $\Delta_b u = 0$) in a complete noncompact pseudohermitian $(2n+1)$-manifold, we need to show the following sub-gradient estimate for positive pseudoharmonic functions u.

Theorem 4. Let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold with

$$(2Ric - (n-2)Tor)\,(Z,\ Z) \geq -2k\,|Z|^2$$

for all $Z \in T_{1,0}$, and $k \geq 0$. Furthermore, we assume that $(M,\ J,\ \theta)$ satisfies the CR sub-Laplacian comparison property (1.5). If $u\,(x)$ is a positive pseudoharmonic function with

$$[\Delta_b, T]u = 0 \tag{1.8}$$

on M. Then for each constant $b>0$, there exists a positive constant $C_2=C_2(k)$ such that

$$\frac{\left|\nabla_b u\right|^2}{u^2}+b\frac{u_0^2}{u^2}<\frac{(n+5+2bk)^2}{(5+2bk)}\left(k+\frac{2}{b}+\frac{C_2}{R}\right) \tag{1.9}$$

on the ball $B\left(R\right)$ of a large enough radius R which depends only on b, k.

Remark 1. In the similar spirit, recently we are able to obtain the CR analogue of matrix Li-Yau-Hamilton Harnack inequality for the positive solution to the CR heat equation in a closed pseudohermitian $(2n+1)$-manifold with nonnegative bisectional curvature and bitorsional tensor. We refer to [CFTW] and [CCF] for more details.

It is shown that (Lemma 11)

$$[\Delta_b,\ T]\,u=4\,\mathrm{Im}[i\sum_{\alpha,\beta=1}^{n}(A_{\bar{\alpha}\bar{\beta}}u_{\beta})_{,\alpha}]. \tag{1.10}$$

If $(M,\ J,\ \theta)$ is a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion. Then

$$[\Delta_b,\ T]\,u=0.$$

Hence, by applying Theorem 3 and Theorem 4, we have

Corollary 5. Let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing pseudohermitian torsion and

$$Ric\left(Z,\ Z\right)\geq -k\left|Z\right|^2$$

for all $Z\in T_{1,0}$, and $k\geq 0$. Let $u\left(x\right)$ be a positive pseudoharmonic function. Then for each constant $b>0$, there exists a positive constant $\overline{C}_2=\overline{C}_2(k)$ such that

$$\frac{\left|\nabla_b u\right|^2}{u^2}+b\frac{u_0^2}{u^2}<\frac{(n+5+2bk)^2}{(5+2bk)}\left(k+\frac{2}{b}+\frac{\overline{C}_2}{R}\right) \tag{1.11}$$

on the ball $B\left(R\right)$ of a large enough radius R which depends only on b, k.

As a consequence, let $R\to\infty$ and then $b\to\infty$ with $k=0$ in (1.11),we have the following CR Liouville-type theorem.

Corollary 6. Let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold of nonnegative pseudohermitian Ricci curvature tensors and vanishing torsion. Then any positive pseudoharmonic function is constant.

Corollary 7. There does not exist any positive nonconstant pseudoharmonic function in a standard Heisenberg $(2n+1)$-manifold $(\mathbf{H}^n,\ \mathbf{J},\ \boldsymbol{\theta})$.

Remark 2. Koranyi and Stanton ([KS]) proved the Liouville theorem in $(\mathbf{H}^n,\ \mathbf{J},\ \boldsymbol{\theta})$ by a different method.

In general, we have the following weak sub-gradient estimate for positive pseudoharmonic functions in a complete noncompact pseudohermitian $(2n+1)$-manifold.

Theorem 8. Let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold with

$$(2Ric-(n-2)Tor)\,(Z,\ Z)\geq -2k\,|Z|^2$$

and

$$\max\left\{|A_{\alpha\beta}|\,,\ |A_{\alpha\beta,\bar{\alpha}}|\right\}\leq k_1 \tag{1.12}$$

for all $Z\in T_{1,0}$ and $k\geq 0,\ k_1>0$. Furthermore, we assume that $(M,\ J,\ \theta)$ satisfies the CR sub-Laplacian comparison property. If $u\,(x)$ is a positive pseudoharmonic function on M. Then there exists a small constant $b_0=b_0(n,k,k_1)>0$ and $C_3=C_4(k,\ k_1,\ k_2)$ such that for any $0<b\leq b_0$,

$$\frac{|\nabla_b u|^2}{u^2}+b\frac{u_0^2}{u^2}<\frac{(n+5)^2}{5}\left(k+n\,(1+b)\,k_1+\frac{2}{b}+\frac{C_3}{R}\right) \tag{1.13}$$

on the ball $B\,(R)$ of a large enough radius R which depends only on b.

Remark 3. By comparing the Yau's gradient estimate (1.1), we need an extra assumption (1.12) to obtain the CR subgradient estimate (1.13) due to the natural of sub-Laplacian in

pseudohermitian geometry. However, we do obtain an extra estimate on the derivative of pseudoharmonic functions $u(x)$ along the missing direction of characteristic vector field T.

We briefly describe the methods used in our proofs. In section 2, we first introduce some basic materials in a pseudohermitian $(2n+1)$-manifold. Then we are able to get the CR Bochner-type estimate and derive some key Lemmas. In section 3, let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold with the CR sub-Laplacian comparison property, we obtain subgradient estimates for positive pseudoharmonic functions. As a consequence, the natural analogue of Liouville-type theorem for the sub-Laplacian holds in a complete noncompact pseudohermitian $(2n+1)$-manifold of nonnegative pseudohermitian Ricci curvature tensor and vanishing torsion. In section 4, we give a proof of sub-Laplacian comparison theorem in a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing pseudohermitian torsion tensors and nonnegative pseudohermitian Ricci curvature tensors. In appendix **A**, we get the CR sub-Laplacian comparison property (1.6) in $(\mathbf{H}^n,\ \mathbf{J},\ \boldsymbol{\theta})$ by a straightforward computation.

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2. The CR Bochner-Type Estimate

In this section, we derive some key lemmas. In particular, we obtain the CR Bochner-type estimate as in Lemma 10.

We first introduce some basic materials in a pseudohermitian $(2n+1)$-manifold (see [L1], [L2] for more details). Let (M,ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with

contact structure ξ. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the following integrability condition: If X and Y are in ξ, then so are $[JX, Y]+[X, JY]$ and $J([JX, Y]+[X, JY]) = [JX, JY]-[X, Y]$.

Let $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ be a frame of $TM\otimes\mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T_{0,1}$ and T is the characteristic vector field. Then $\left\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\right\}$, which is the coframe dual to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$, satisfies

$$d\theta = ih_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} \tag{2.1}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar{\beta}})$. Actually we can always choose Z_α such that $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$; hence, throughout this note, we assume $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$.

The Levi form $\langle\ ,\ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by

$$\langle Z, W\rangle_{L_\theta} = -i\left\langle d\theta, Z\wedge\overline{W}\right\rangle . \tag{2.2}$$

We can extend $\langle\ ,\ \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\left\langle\overline{Z}, \overline{W}\right\rangle_{L_\theta} = \overline{\langle Z, W\rangle}_{L_\theta}$ for all $Z, W \in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle\ ,\ \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over M with respect to the volume form $d\mu = \theta\wedge(d\theta)^n$, we get an inner product on the space of sections of each tensor bundle. We denote the inner product by the notation $\langle\ ,\ \rangle$. For example

$$\langle u, v\rangle = \int_M u\overline{v}\ d\mu,$$

for functions u and v.

The pseudohermitian connection of (J, θ) is the connection ∇ on $TM\otimes\mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha \in T_{1,0}$ by

$$\nabla Z_\alpha = \theta_\alpha{}^\beta \otimes Z_\beta, \quad \nabla Z_{\bar{\alpha}} = \theta_{\bar{\alpha}}{}^{\bar{\beta}} \otimes Z_{\bar{\beta}}, \quad \nabla T = 0,$$

where $\theta_\alpha{}^\beta$ are the 1-forms uniquely determined by the following equations:

$$
\begin{aligned}
d\theta^{\beta} &= \theta^{\alpha} \wedge \theta_{\alpha}{}^{\beta} + \theta \wedge \tau^{\beta}, \\
0 &= \tau_{\alpha} \wedge \theta^{\alpha}, \\
0 &= \theta_{\alpha}{}^{\beta} + \theta_{\bar{\beta}}{}^{\bar{\alpha}},
\end{aligned} \tag{2.3}
$$

We can write (by Cartan lemma) $\tau_{\alpha} = A_{\alpha\gamma}\theta^{\gamma}$ with $A_{\alpha\gamma} = A_{\gamma\alpha}$. The curvature of Tanaka-Webster connection, expressed in terms of the coframe $\{\theta = \theta^{0}, \theta^{\alpha}, \theta^{\bar{\alpha}}\}$, is

$$
\begin{aligned}
\Pi_{\beta}{}^{\alpha} &= \overline{\Pi_{\bar{\beta}}{}^{\bar{\alpha}}} = d\omega_{\beta}{}^{\alpha} - \omega_{\beta}{}^{\gamma} \wedge \omega_{\gamma}{}^{\alpha}, \\
\Pi_{0}{}^{\alpha} &= \Pi_{\alpha}{}^{0} = \Pi_{0}{}^{\bar{\beta}} = \Pi_{\bar{\beta}}{}^{0} = \Pi_{0}{}^{0} = 0.
\end{aligned}
$$

Webster showed that $\Pi_{\beta}{}^{\alpha}$ can be written

$$
\Pi_{\beta}{}^{\alpha} = R_{\beta}{}^{\alpha}{}_{\rho\bar{\sigma}}\theta^{\rho} \wedge \theta^{\bar{\sigma}} + W_{\beta}{}^{\alpha}{}_{\rho}\theta^{\rho} \wedge \theta - W^{\alpha}{}_{\beta\bar{\rho}}\theta^{\bar{\rho}} \wedge \theta + i\theta_{\beta} \wedge \tau^{\alpha} - i\tau_{\beta} \wedge \theta^{\alpha} \tag{2.4}
$$

where the coefficients satisfy

$$
R_{\beta\bar{\alpha}\rho\bar{\sigma}} = \overline{R_{\alpha\bar{\beta}\sigma\bar{\rho}}} = R_{\bar{\alpha}\beta\bar{\sigma}\rho} = R_{\rho\bar{\alpha}\beta\bar{\sigma}}, \quad W_{\beta\bar{\alpha}\gamma} = W_{\gamma\bar{\alpha}\beta}.
$$

We will denote components of covariant derivatives with indices preceded by a comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0, \alpha, \bar{\alpha}\}$ indicate derivatives with respect to $\{T, Z_{\alpha}, Z_{\bar{\alpha}}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $u_{\alpha} = Z_{\alpha}u,\ u_{\alpha\bar{\beta}} = Z_{\bar{\beta}}Z_{\alpha}u - \omega_{\alpha}{}^{\gamma}(Z_{\bar{\beta}})Z_{\gamma}u$.

For a real function u, the subgradient ∇_b is defined by $\nabla_b u \in \xi$ and $\langle Z, \nabla_b u\rangle_{L_\theta} = du(Z)$ for all vector fields Z tangent to contact plane. Locally $\nabla_b u = \sum_{\alpha} u_{\bar{\alpha}}Z_{\alpha} + u_{\alpha}Z_{\bar{\alpha}}$. We can use the connection to define the subhessian as the complex linear map

$$
(\nabla^{H})^{2}u : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1}
$$

by

$$
(\nabla^{H})^{2}u(Z) = \nabla_{Z}\nabla_{b}u.
$$

In particular,

$$|\nabla_b u|^2 = 2u_\alpha u_{\overline{\alpha}}, \quad |\nabla_b^2 u|^2 = 2(u_{\alpha\beta}u_{\overline{\alpha}\overline{\beta}} + u_{\alpha\overline{\beta}}u_{\overline{\alpha}\beta}).$$

Also

$$\Delta_b u = Tr\left((\nabla^H)^2 u\right) = \textstyle\sum_\alpha (u_{\alpha\bar{\alpha}} + u_{\bar{\alpha}\alpha}).$$

Next we recall the following commutation relations ([L1]). Let φ be a scalar function and $\sigma = \sigma_\alpha \theta^\alpha$ be a $(1,0)$ form, then we have

$$\begin{aligned}
\varphi_{\alpha\beta} &= \varphi_{\beta\alpha}, \\
\varphi_{\alpha\bar{\beta}} - \varphi_{\bar{\beta}\alpha} &= ih_{\alpha\overline{\beta}}\varphi_0, \\
\varphi_{0\alpha} - \varphi_{\alpha 0} &= A_{\alpha\beta}\varphi_{\bar{\beta}}, \\
\sigma_{\alpha,0\beta} - \sigma_{\alpha,\beta 0} &= \sigma_{\alpha,\bar{\gamma}}A_{\gamma\beta} - \sigma_\gamma A_{\alpha\beta,\bar{\gamma}}, \\
\sigma_{\alpha,0\bar{\beta}} - \sigma_{\alpha,\bar{\beta}0} &= \sigma_{\alpha,\gamma}A_{\bar{\gamma}\bar{\beta}} + \sigma_\gamma A_{\bar{\gamma}\bar{\beta},\alpha},
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
\sigma_{\alpha,\beta\gamma} - \sigma_{\alpha,\gamma\beta} &= iA_{\alpha\gamma}\sigma_\beta - iA_{\alpha\beta}\sigma_\gamma, \\
\sigma_{\alpha,\bar{\beta}\bar{\gamma}} - \sigma_{\alpha,\bar{\gamma}\bar{\beta}} &= ih_{\alpha\overline{\beta}}A_{\bar{\gamma}\bar{\rho}}\sigma_\rho - ih_{\alpha\overline{\gamma}}A_{\bar{\beta}\bar{\rho}}\sigma_\rho, \\
\sigma_{\alpha,\beta\bar{\gamma}} - \sigma_{\alpha,\bar{\gamma}\beta} &= ih_{\beta\overline{\gamma}}\sigma_{\alpha,0} + R_{\alpha\bar{\rho}\beta\bar{\gamma}}\sigma_\rho.
\end{aligned} \tag{2.6}$$

Now we recall a lemma from A. Greenleaf ([Gr]) and also ([CC2]).

Lemma 9. For a real function φ,

$$\begin{aligned}
\Delta_b |\nabla_b\varphi|^2 &= 2\left|\left(\nabla^H\right)^2\varphi\right|^2 + 2\left\langle \nabla_b\varphi,\ \nabla_b\Delta_b\varphi\right\rangle \\
&+ \ (4Ric - 2\,(n-2)\,Tor)\left((\nabla_b\varphi)_C\,,\ \ (\nabla_b\varphi)_C\right) + 4\left\langle J\nabla_b\varphi,\ \nabla_b\varphi_0\right\rangle,
\end{aligned} \tag{2.7}$$

where $(\nabla_b\varphi)_C = \varphi_{\bar{\alpha}}Z_\alpha$ is the corresponding complex $(1,\ 0)$-vector of $\nabla_b\varphi$.

Lemma 10. For a smooth real-valued function φ and any $\nu > 0$, we have

$$\begin{aligned}
\Delta_b |\nabla_b\varphi|^2 \ &\geq \ 4\left(\sum_{\alpha,\beta=1}^{n}\left|\varphi_{a\beta}\right|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|\varphi_{a\bar{\beta}}\right|^2\right) + \tfrac{1}{n}\left(\Delta_b\varphi\right)^2 + n\varphi_0^2 + 2\left\langle \nabla_b\varphi,\ \nabla_b\Delta_b\varphi\right\rangle \\
&+ \ \left(4Ric - 2\,(n-2)\,Tor - \tfrac{4}{\nu}\right)\left((\nabla_b\varphi)_C\,,\ \ (\nabla_b\varphi)_C\right) - 2\nu\left|\nabla_b\varphi_0\right|^2.
\end{aligned}$$

Proof. Since

$$\begin{aligned} |(\nabla^H)^2\varphi|^2 &= 2\textstyle\sum_{\alpha,\beta=1}^n(\varphi_{\alpha\beta}\varphi_{\overline{\alpha}\overline{\beta}} + \varphi_{\alpha\overline{\beta}}\varphi_{\overline{\alpha}\beta}) \\ &= 2\textstyle\sum_{\alpha,\beta=1}^n(|\varphi_{\alpha\beta}|^2 + |\varphi_{\alpha\overline{\beta}}|^2) \\ &= 2(\textstyle\sum_{\alpha,\beta=1}^n|\varphi_{\alpha\beta}|^2 + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^n|\varphi_{\alpha\overline{\beta}}|^2 + \sum_{\alpha=1}^n|\varphi_{\alpha\overline{\alpha}}|^2) \end{aligned}$$

and from the commutation relation (2.5)

$$\begin{aligned} \textstyle\sum_{\alpha=1}^n|\varphi_{\alpha\overline{\alpha}}|^2 &= \tfrac{1}{4}\textstyle\sum_{\alpha=1}^n\left(|\varphi_{\alpha\overline{\alpha}} + \varphi_{\overline{\alpha}\alpha}|^2 + \varphi_0^2\right) \\ &= \tfrac{1}{4}\textstyle\sum_{\alpha=1}^n|\varphi_{\alpha\overline{\alpha}} + \varphi_{\overline{\alpha}\alpha}|^2 + \tfrac{n}{4}\varphi_0^2. \end{aligned}$$

It follows that

$$\begin{aligned} |(\nabla^H)^2\varphi|^2 &= 2(\textstyle\sum_{\alpha,\beta=1}^n|\varphi_{\alpha\beta}|^2 + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^n|\varphi_{\alpha\overline{\beta}}|^2) + \tfrac{1}{2}\sum_{\alpha=1}^n|\varphi_{\alpha\overline{\alpha}} + \varphi_{\overline{\alpha}\alpha}|^2 + \tfrac{n}{2}\varphi_0^2 \\ &\leq 2(\textstyle\sum_{\alpha,\beta=1}^n|\varphi_{\alpha\beta}|^2 + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^n|\varphi_{\alpha\overline{\beta}}|^2) + \tfrac{1}{2n}\left(\Delta_b\varphi\right)^2 + \tfrac{n}{2}\varphi_0^2. \end{aligned}$$

On the other hand, for all $\nu > 0$

$$\begin{aligned} 4\left\langle J\nabla_b\varphi,\ \nabla_b\varphi_0\right\rangle &\geq -4\left|\nabla_b\varphi\right|\left|\nabla_b\varphi_0\right| \\ &\geq -\tfrac{2}{\nu}\left|\nabla_b\varphi\right|^2 - 2\nu\left|\nabla_b\varphi_0\right|^2. \end{aligned}$$

Then the result follows easily from Lemma 9. □

Definition 3. ([GL]) Let (M, J, θ) be a pseudohermitian $(2n+1)$-manifold. We define the purely holomorphic second-order operator Q by

$$Q\varphi = 2i\sum_{\alpha,\beta=1}^n (A_{\bar{\alpha}\bar{\beta}}\varphi_\beta)_{,\alpha}\,.$$

By apply the commutation relations (2.5), one obtains

Lemma 11. ([GL], [CKL]) Let $\varphi\left(x\right)$ be a smooth function defined on M. Then

$$\Delta_b\varphi_0 = \left(\Delta_b\varphi\right)_0 + 2\sum_{\alpha,\beta=1}^n\left[\left(A_{\alpha\beta}\varphi_{\bar{\beta}}\right)_{\bar{\alpha}} + \left(A_{\bar{\alpha}\bar{\beta}}\varphi_\beta\right)_\alpha\right].$$

That is

$$2\,\mathrm{Im}\,Q\varphi = \left[\Delta_b,\ T\right]\varphi.$$

Proof. By direct computation and the commutation relation (2.5), we have

$$
\begin{aligned}
\Delta_b\varphi_0 &= \varphi_{0\alpha\overline{\alpha}}+\varphi_{0\overline{\alpha}\alpha}\\
&= \left(\varphi_{\alpha 0}+A_{\alpha\beta}\varphi_{\overline{\beta}}\right)_{\overline{\alpha}}+\text{conjugate}\\
&= \varphi_{\alpha 0\overline{\alpha}}+\left(A_{\alpha\beta}\varphi_{\overline{\beta}}\right)_{\overline{\alpha}}+\text{conjugate}\\
&= \varphi_{\alpha\overline{\alpha}0}+\varphi_{\overline{\alpha}\alpha 0}+2\left[\left(A_{\alpha\beta}\varphi_{\overline{\beta}}\right)_{\overline{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\right)_{\alpha}\right]\\
&= \left(\Delta_b\varphi\right)_0+2\left[\left(A_{\alpha\beta}\varphi_{\overline{\beta}}\right)_{\overline{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\right)_{\alpha}\right].
\end{aligned}
$$

This completes the proof. □

Let u be a positive pseudoharmonic function and $f(x)=\ln u(x)$. Then

$$\Delta_b f=-\left|\nabla_b f\right|^2.$$

We first define

$$V(\varphi)=\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}\varphi_{\bar{\beta}}\right)_{\bar{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\right)_{\alpha}+A_{\alpha\beta}\varphi_{\bar{\beta}}\varphi_{\bar{\alpha}}+A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\varphi_{\alpha}\right].$$

Lemma 12. Let u be a positive pseudoharmonic function with $f=\ln u$. Then

$$\Delta_b f_0=-2\left\langle\nabla_b f,\ \nabla_b f_0\right\rangle+2V(f).$$

Proof. From Lemma 11

$$\Delta_b f_0=\left(\Delta_b f\right)_0+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}\varphi_{\bar{\beta}}\right)_{\bar{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\right)_{\alpha}\right].$$

Since

$$\Delta_b f=-\left|\nabla_b f\right|^2,$$

it follows from the commutation relation (2.5) that

$$
\begin{aligned}
\Delta_b f_0 &= \left(\Delta_b f\right)_0+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar{\beta}}\right)_{\bar{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}f_{\beta}\right)_{\alpha}\right]\\
&= \left(-\left|\nabla_b f\right|^2\right)_0+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar{\beta}}\right)_{\bar{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}f_{\beta}\right)_{\alpha}\right]\\
&= -2\left\langle\nabla_b f_0,\ \nabla_b f\right\rangle+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar{\beta}}\right)_{\bar{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}f_{\beta}\right)_{\alpha}+A_{\alpha\beta}f_{\bar{\alpha}}f_{\bar{\beta}}+A_{\bar{\alpha}\bar{\beta}}f_{\alpha}f_{\beta}\right].
\end{aligned}
$$

□

Lemma 13. Let (M, J, θ) be a pseudohermitian $(2n+1)$-manifold and u be a positive function with $f = \ln u$. Suppose that

$$2\operatorname{Im} Qu = [\Delta_b,\ T]\, u = 0.$$

Then

$$V(f) = 0. \tag{2.8}$$

Proof. We compute

$$\begin{aligned}
V(f) &= \sum_{\alpha,\beta=1}^{n} \left[\left(A_{\alpha\beta} f_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} f_{\beta}\right)_{\alpha} + A_{\alpha\beta} f_{\bar\alpha} f_{\bar\beta} + A_{\bar\alpha\bar\beta} f_{\alpha} f_{\beta} \right] \\
&= \sum_{\alpha,\beta=1}^{n} \left[A_{\alpha\beta} f_{\bar\beta\bar\alpha} + A_{\alpha\beta,\bar\alpha} f_{\bar\beta} + A_{\bar\alpha\bar\beta} f_{\beta\alpha} + A_{\bar\alpha\bar\beta,\alpha} f_{\beta} + A_{\alpha\beta} f_{\bar\alpha} f_{\bar\beta} + A_{\bar\alpha\bar\beta} f_{\alpha} f_{\beta} \right] \\
&= \sum_{\alpha,\beta=1}^{n} \Big\{ A_{\bar\alpha\bar\beta} \left(\tfrac{u_{\beta\alpha}}{u} - \tfrac{u_\alpha u_\beta}{u^2} \right) + A_{\alpha\beta} \left(\tfrac{u_{\bar\beta\bar\alpha}}{u} - \tfrac{u_{\bar\alpha} u_{\bar\beta}}{u^2} \right) \\
&\quad + A_{\bar\alpha\bar\beta,\alpha} \tfrac{u_\beta}{u} + A_{\alpha\beta,\bar\alpha} \tfrac{u_{\bar\beta}}{u} + A_{\bar\alpha\bar\beta} \tfrac{u_\alpha u_\beta}{u^2} + A_{\alpha\beta} \tfrac{u_{\bar\alpha} u_{\bar\beta}}{u^2} \Big\} \\
&= \sum_{\alpha,\beta=1}^{n} \tfrac{1}{u} \left[\left(A_{\alpha\beta} u_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} u_{\beta}\right)_{\alpha} \right] \\
&= \tfrac{1}{2u} [\Delta_b,\ T]\, u.
\end{aligned} \tag{2.9}$$

This completes the proof. ∎

3. CR Analogue of Yau's Gradient Estimate

In this section, we will prove main Theorem 8 and Theorem 4. We first recall a real-valued function

$$F(x,\ t,\ R,\ b) : M \times [0,\ 1] \times (0,\ \infty) \times (0,\ \infty) \to \mathbb{R}$$

defined by

$$F(x,\ t,\ R,\ b) = t \left(|\nabla_b f|^2 (x) + bt\eta(x)\, f_0^2(x) \right), \tag{3.1}$$

where $\eta(x) : M \to [0,\ 1]$ is a smooth cut-off function defined by

$$\eta(x) = \eta(r(x)) = \begin{cases} 1, & x \in\ B(R) \\ 0, & x \in M \backslash B(2R) \end{cases}$$

such that

$$-\frac{C}{R}\eta^{\frac{1}{2}} \le \eta' \le 0 \tag{3.2}$$

and

$$\left|\eta''\right| \le \frac{C}{R^2}, \tag{3.3}$$

where we denote $\frac{\partial}{\partial r}\eta$ by η' and $r(x)$ is the Carnot-Carathéodory distance to a fixed point x_0.

In the following calculation, the universal constant C might be changed from lines to lines.

Proposition 14. Let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold with

$$(2Ric - (n-2)\,Tor)\,(Z,\ Z) \ge -2k\,|Z|^2 \tag{3.4}$$

for all $Z \in T_{1,0}$, where k is an nonnegative constant. Suppose that $(M,\ J,\ \theta)$ satisfies the CR sub-Laplacian comparison property. Then

$$\begin{array}{rcl}\Delta_b F & \ge & -2\left\langle \nabla_b f,\ \nabla_b F\right\rangle + t\left[4\sum\limits_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2 + 4\sum\limits_{\alpha,\beta=1,\alpha\ne\beta}^{n}\left|f_{a\bar{\beta}}\right|^2 + \frac{1}{n}\left(\Delta_b f\right)^2\right. \\ & & \left. + \left(n - \frac{bCt}{R}\right)f_0^2 - \left(2k + \frac{4}{bt\eta}\right)\left|\nabla_b f\right|^2 - \frac{bCt}{R}\eta\left|\nabla_b f\right|^2 f_0^2 + 4bt\eta f_0 V(f)\right].\end{array}$$

Proof. By CR sub-Laplacian comparison property,

$$\begin{array}{rcl}\Delta_b\eta & = & \eta'' + \eta'\Delta_b r \\ & \ge & -\frac{C}{R^2} - \frac{C}{R}\left(\frac{C_1}{R} + C_2\right) \\ & \ge & -\frac{C}{R}.\end{array}$$

First we compute

$$\begin{array}{rcl}\Delta_b\left(bt\eta f_0^2\right) & = & bt\left(f_0^2\Delta_b\eta + \eta\Delta_b f_0^2 + 2\left\langle \nabla_b\eta,\ \nabla_b f_0^2\right\rangle\right) \\ & \ge & bt\left(-\frac{C}{R}f_0^2 + 2\eta f_0\Delta_b f_0 + 2\eta\left|\nabla_b f_0\right|^2 + 4f_0\left\langle \nabla_b\eta,\ \nabla_b f_0\right\rangle\right) \\ & \ge & bt\left(-\frac{C}{R}f_0^2 + 2\eta f_0\Delta_b f_0 + 2\eta\left|\nabla_b f_0\right|^2 - 4\left|f_0\right|\left|\nabla_b\eta\right|\left|\nabla_b f_0\right|\right) \\ & \ge & bt[-\frac{C}{R}f_0^2 + 2\eta f_0\Delta_b f_0 + \left(2\eta - 2\cdot\frac{1}{2}\eta\right)\left|\nabla_b f_0\right|^2 - 2\cdot 2\eta^{-1}\left|\nabla_b\eta\right|^2 f_0^2] \\ & \ge & bt[-\frac{C}{R}f_0^2 + 2\eta f_0\Delta_b f_0 + \left(2\eta - 2\cdot\frac{1}{2}\eta\right)\left|\nabla_b f_0\right|^2],\end{array} \tag{3.5}$$

where we use the Young's inequality and the inequality (3.2) which implies that

$$\eta^{-1}\left|\nabla_b\eta\right|^2 \le \frac{C}{R^2}.$$

Second, it follows from assumption (3.4), Lemma 10 and (3.5) that

$$\begin{aligned}
\Delta_b F &= t\left(\Delta_b\left|\nabla_b f\right|^2+\Delta_b\left(bt\eta f_0^2\right)\right)\\
&\geq t\Bigg(4\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2+4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar\beta}\right|^2+\tfrac{1}{n}\left(\Delta_b f\right)^2+nf_0^2+2\left\langle\nabla_b f,\ \nabla_b\Delta_b f\right\rangle\\
&\qquad -2\left(k+\tfrac{1}{\nu}\right)\left|\nabla_b f\right|^2-2\nu\left|\nabla_b f_0\right|^2-\tfrac{bCt}{R}f_0^2+2bt\eta f_0\Delta_b f_0+2\cdot\tfrac{bt}{2}\eta\left|\nabla_b f_0\right|^2\Bigg)\\
&\geq t\Bigg[4\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2+4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar\beta}\right|^2+\tfrac{1}{n}\left(\Delta_b f\right)^2+\left(n-\tfrac{bCt}{R}\right)f_0^2+2\left\langle\nabla_b f,\ \nabla_b\Delta_b f\right\rangle\\
&\qquad -2\left(k+\tfrac{1}{\nu}\right)\left|\nabla_b f\right|^2+2\left(\tfrac{bt}{2}\eta-\nu\right)\left|\nabla_b f_0\right|^2+2bt\eta f_0\Delta_b f_0\Bigg].
\end{aligned}$$

Then taking $\nu=\frac{bt\eta}{2}$,

$$\begin{aligned}
\Delta_b F \ \geq\ & t\Bigg[4\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2+4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar\beta}\right|^2+\tfrac{1}{n}\left(\Delta_b f\right)^2+\left(n-\tfrac{bCt}{R}\right)f_0^2\\
&-2\left(k+\tfrac{2}{bt\eta}\right)\left|\nabla_b f\right|^2+2\left\langle\nabla_b f,\ \nabla_b\Delta_b f\right\rangle+2bt\eta f_0\Delta_b f_0\Bigg].
\end{aligned}\tag{3.6}$$

Finally, by Lemma 12

$$\begin{aligned}
&2\left\langle\nabla_b f,\ \nabla_b\Delta_b f\right\rangle+2bt\eta f_0\Delta_b f_0\\
=\ &2\left\langle\nabla_b f,\ \nabla_b\left(-\left|\nabla_b f\right|^2\right)\right\rangle+2bt\eta f_0\left[-2\left\langle\nabla_b f,\ \nabla_b f_0\right\rangle+2V(f)\right]\\
=\ &-2\left\langle\nabla_b f,\ \nabla_b\left(\tfrac{F}{t}-bt\eta f_0^2\right)\right\rangle-4bt\eta f_0\left\langle\nabla_b f,\ \nabla_b f_0\right\rangle+4bt\eta f_0V(f)\\
=\ &-\tfrac{2}{t}\left\langle\nabla_b f,\ \nabla_b F\right\rangle+2bt\left\langle\nabla_b f,\ \nabla_b\left(\eta f_0^2\right)\right\rangle-4bt\eta f_0\left\langle\nabla_b f,\ \nabla_b f_0\right\rangle+4bt\eta f_0V(f)\\
=\ &-\tfrac{2}{t}\left\langle\nabla_b f,\ \nabla_b F\right\rangle+2btf_0^2\left\langle\nabla_b f,\ \nabla_b\eta\right\rangle+4bt\eta f_0V(f)
\end{aligned}$$

Now by Young's inequality, we have

$$\begin{aligned}
&2\left\langle\nabla_b f,\ \nabla_b\Delta_b f\right\rangle+2bt\eta f_0\Delta_b f_0\\
=\ &-\tfrac{2}{t}\left\langle\nabla_b f,\ \nabla_b F\right\rangle+2btf_0^2\left\langle\nabla_b f,\ \nabla_b\eta\right\rangle+4bt\eta f_0V(f)\\
\geq\ &-\tfrac{2}{t}\left\langle\nabla_b f,\ \nabla F\right\rangle-2btf_0^2\left|\nabla_b f\right|\left|\nabla_b\eta\right|+4bt\eta f_0V(f)\\
\geq\ &-\tfrac{2}{t}\left\langle\nabla_b f,\ \nabla F\right\rangle-\tfrac{2Cbt}{R}f_0^2\left|\nabla_b f\right|\eta^{\frac{1}{2}}+4bt\eta f_0V(f)\\
\geq\ &-\tfrac{2}{t}\left\langle\nabla_b f,\ \nabla F\right\rangle-\tfrac{Cbt}{R}f_0^2-\tfrac{Cbt}{R}\eta f_0^2\left|\nabla_b f\right|^2+4bt\eta f_0V(f).
\end{aligned}\tag{3.7}$$

Substituting (3.7) into (3.6),

$$\begin{aligned}\Delta_b F \;\geq\; & -2\left\langle \nabla_b f,\ \nabla F\right\rangle + t\left[4\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2 + 4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2 + \tfrac{1}{n}\left(\Delta_b f\right)^2\right.\\ & \left. +\left(n-\tfrac{bCt}{R}\right)f_0^2 - 2\left(k+\tfrac{2}{bt\eta}\right)\left|\nabla_b f\right|^2 - \tfrac{Cbt}{R}\eta f_0^2\left|\nabla_b f\right|^2 + 4bt\eta f_0 V(f)\right].\end{aligned}$$

□

Proposition 15. Let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold with

$$\left(2Ric-(n-2)\,Tor\right)(Z,\ Z)\geq -2k\left|Z\right|^2$$

for all $Z\in T_{1,0}$, where k is an nonnegative constant. Suppose that $(M,\ J,\ \theta)$ satisfies the CR sub-Laplacian comparison property. Then for all $a\neq 0$

$$\begin{aligned}t\eta\Delta_b(\eta F) \;\geq\; & \tfrac{1}{na^2}(\eta F)^2 - \tfrac{C}{R}(\eta F) + 2t\eta\left\langle \nabla_b\eta,\ \nabla_b F\right\rangle - 2t\eta^2\left\langle \nabla_b f,\ \nabla_b F\right\rangle\\ & +4t^2\eta^2\left(\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2+\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2\right)\\ & +\left[n-\tfrac{bC}{R}-\left(\tfrac{2b}{na^2}+\tfrac{bC}{R}\right)(\eta F)\right]t^2\eta^2 f_0^2\\ & +\left[-\tfrac{2(1+a)}{na^2}(\eta F)-2k-\tfrac{4}{b}\right]t\eta\left|\nabla_b f\right|^2 + 4bt^3\eta^3 f_0 V(f).\end{aligned}\tag{3.8}$$

Proof. By using Proposition 14, we first compute

$$\begin{aligned}\Delta_b(\eta F) \;=\; & (\Delta_b\eta)F + 2\left\langle \nabla_b\eta,\ \nabla_b F\right\rangle + \eta\Delta_b F\\ \geq\; & -\tfrac{C}{R}F + 2\left\langle \nabla_b\eta,\ \nabla_b F\right\rangle - 2\eta\left\langle \nabla_b f,\ \nabla F\right\rangle\\ & +t\eta\left[4\left(\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2+\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2\right)+\tfrac{1}{n}\left(\Delta_b f\right)^2+\left(n-\tfrac{bCt}{R}\right)f_0^2\right.\\ & \left.-2\left(k+\tfrac{2}{bt\eta}\right)\left|\nabla_b f\right|^2 - \tfrac{Cbt}{R}\eta f_0^2\left|\nabla_b f\right|^2 + 4bt\eta f_0 V(f)\right]\end{aligned}$$

and for each $a \neq 0$

$$\begin{aligned}
(\Delta_b f)^2 &= \left(-|\nabla_b f|^2\right)^2 \\
&= \left(\tfrac{1}{at}F - \tfrac{1}{a}|\nabla_b f|^2 - \tfrac{1}{a}bt\eta f_0^2 - |\nabla_b f|^2\right)^2 \\
&= \left(\tfrac{1}{at}F - \tfrac{a+1}{a}|\nabla_b f|^2 - \tfrac{1}{a}bt\eta f_0^2\right)^2 \\
&= \tfrac{1}{a^2t^2}F^2 + \left(\tfrac{a+1}{a}\right)^2|\nabla_b f|^4 + \tfrac{1}{a^2}b^2t^2\eta^2 f_0^4 \\
&\quad -\tfrac{2(a+1)}{a^2t}F|\nabla_b f|^2 - \tfrac{2b}{a^2}\eta F f_0^2 + \tfrac{2(a+1)bt}{a^2}\eta|\nabla_b f|^2 f_0^2 \\
&\geq \tfrac{1}{a^2t^2}F^2 - \tfrac{2(a+1)}{a^2t}F|\nabla_b f|^2 - \tfrac{2b}{a^2}\eta F f_0^2.
\end{aligned}$$

Then

$$\begin{aligned}
\Delta_b(\eta F) \geq\ & \tfrac{1}{na^2t}\eta F^2 - \tfrac{C}{R}F + 2\langle\nabla_b\eta,\ \nabla_b F\rangle - 2\eta\langle\nabla_b f,\ \nabla F\rangle \\
&+4t\eta\left(\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2\right) \\
&+\left(n - \tfrac{bCt}{R} - \tfrac{2b}{na^2}\eta F\right)t\eta f_0^2 + \left(-\tfrac{2(1+a)}{na^2}\eta F - 2kt\eta - \tfrac{4}{b}\right)|\nabla_b f|^2 \\
&-\tfrac{Cb}{R}\left(t\eta|\nabla_b f|^2\right)\left(t\eta f_0^2\right) + 4bt^2\eta^2 f_0 V(f).
\end{aligned}$$

Hence

$$\begin{aligned}
\Delta_b(\eta F) \geq\ & \tfrac{1}{na^2t}\eta F^2 - \tfrac{C}{R}F + 2\langle\nabla_b\eta,\ \nabla_b F\rangle - 2\eta\langle\nabla_b f,\ \nabla F\rangle \\
&+4t\eta\left(\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2\right) \\
&+\left[n - \tfrac{bCt}{R} - \left(\tfrac{2b}{na^2} + \tfrac{bC}{R}\right)\eta F\right]t\eta f_0^2 \\
&+\left(-\tfrac{2(1+a)}{na^2}\eta F - 2kt\eta - \tfrac{4}{b}\right)|\nabla_b f|^2 + 4bt^2\eta^2 f_0 V(f).
\end{aligned} \tag{3.9}$$

Finally, multiply $t\eta$ on the both sides of (3.9) and note that $t \leq 1$, $\eta \leq 1$

$$\begin{aligned}
t\eta\Delta_b(\eta F) \geq\ & \tfrac{1}{na^2}(\eta F)^2 - \tfrac{C}{R}\eta F + 2t\eta\langle\nabla_b\eta,\ \nabla_b F\rangle - 2t\eta^2\langle\nabla_b f,\ \nabla_b F\rangle \\
&+4t^2\eta^2\left(\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2\right) \\
&+\left[n - \tfrac{bC}{R} - \left(\tfrac{2b}{na^2} + \tfrac{bC}{R}\right)(\eta F)\right]t^2\eta^2 f_0^2 \\
&+\left[-\tfrac{2(1+a)}{na^2}(\eta F) - 2k - \tfrac{4}{b}\right]t\eta|\nabla_b f|^2 + 4bt^3\eta^3 f_0 V(f).
\end{aligned}$$

$\square$

Proposition 16. Let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold with

$$\left(2Ric-(n-2)\,Tor\right)(Z,\ Z)\geq -2k\left|Z\right|^{2}$$

for all $Z\in T_{1,0}$, where k is an nonnegative constant. Suppose that $(M,\ J,\ \theta)$ satisfies the CR sub-Laplacian comparison property. Let $b,\ R$ be fixed, and $p(t)\in B(2R)$ be the maximal point of ηF for each $t\in(0,1]$. Then at $(\ p(t)\,,t)$ we have

$$\begin{array}{rcl} 0 & \geq & \left(\frac{1}{na^{2}}-\frac{C}{R}\right)(\eta F)^{2}-\frac{3C}{R}(\eta F)+4t^{2}\eta^{2}\left(\sum\limits_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^{2}+\sum\limits_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^{2}\right)\\ & & +\left[n-\frac{bC}{R}-\left(\frac{2b}{na^{2}}+\frac{bC}{R}\right)(\eta F)\right]t^{2}\eta^{2}f_{0}^{2}\\ & & +\left[-\frac{2(1+a)}{na^{2}}(\eta F)-2k-\frac{4}{b}-\frac{C}{R}\right]t\eta\left|\nabla_{b}f\right|^{2}+4bt^{3}\eta^{3}f_{0}V(f)\,. \end{array} \tag{3.10}$$

Proof. Since $(\eta F)(p(t)\,,\ t,\ R,\ b)=\max\limits_{x\in B(2R)}(\eta F)(x,\ t,\ R,\ b)$, at a critical point $(p(t)\,,\ t)$ of $(\eta F)(x,\ t,\ R,\ b)$, we have

$$\nabla_{b}(\eta F)(p(t)\,,\ t,\ R,\ b)=0.$$

This implies that

$$F\nabla_{b}\eta+\eta\nabla_{b}F=0 \tag{3.11}$$

at $(p(t)\,,\ t)$. On the other hand,

$$\Delta_{b}(\eta F)(p(t)\,,\ t,\ R,\ b)\leq 0 \tag{3.12}$$

at $(p(t)\,,\ t)$.

Now we apply (3.11) to $2t\eta\left\langle\nabla_{b}\eta,\ \nabla_{b}F\right\rangle$ and $-2t\eta^{2}\left\langle\nabla_{b}f,\ \nabla_{b}F\right\rangle$ in (3.8), we can derive the following estimates.

$$\begin{array}{rcl} 2t\eta\left\langle\nabla_{b}\eta,\ \nabla_{b}F\right\rangle & = & -2tF\left|\nabla_{b}\eta\right|^{2}\\ & \geq & -\frac{2tC}{R^{2}}\eta F\\ & \geq & -\frac{2C}{R}\eta F \end{array} \tag{3.13}$$

and

$$
\begin{aligned}
-2t\eta^2 \left\langle \nabla_b f,\ \nabla_b F\right\rangle &= 2t\eta F \left\langle \nabla_b f,\ \nabla_b \eta\right\rangle \\
&\geq -2t\left(\eta F\right)\left|\nabla_b f\right| \left|\ \nabla_b \eta\right| \\
&\geq -\tfrac{2tC}{R}\left(\eta F\right)\eta^{\frac{1}{2}}\left|\nabla_b f\right| \\
&\geq -\tfrac{Ct}{R}\left(\eta F\right)^2 - \tfrac{C}{R}t\eta\left|\nabla_b f\right|^2 .
\end{aligned} \tag{3.14}
$$

Here we have applied the Young's inequality for (3.14).

Finally, substituting (3.12), (3.13) and (3.14) into (3.8) in Proposition 15, and noting that $t \leq 1$,

$$
\begin{aligned}
0 \geq\ & \left(\tfrac{1}{na^2} - \tfrac{C}{R}\right)\left(\eta F\right)^2 - \tfrac{3C}{R}\left(\eta F\right) + 4t^2\eta^2\left(\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2\right) \\
& + \left[n - \tfrac{bC}{R} - \left(\tfrac{2b}{na^2} + \tfrac{bC}{R}\right)\left(\eta F\right)\right]t^2\eta^2 f_0^2 \\
& + \left[-\tfrac{2(1+a)}{na^2}\left(\eta F\right) - 2k - \tfrac{4}{b} - \tfrac{C}{R}\right]t\eta\left|\nabla_b f\right|^2 + 4bt^3\eta^3 f_0 V\left(f\right).
\end{aligned}
$$

This completes the proof. □

Now, we are ready to prove our main theorems.

Proof of Theorem 4 :

Proof. We observe that

$$
V\left(f\right) = 0 \tag{3.15}
$$

by assumption (1.8) and Lemma 13.

We begin by substituting (3.15) into (3.10) in Proposition 16 at the maximum point $(p(t), t)$. Hence

$$
\begin{aligned}
0 \geq\ & \left(\tfrac{1}{na^2} - \tfrac{C}{R}\right)\left[\left(\eta F\right)\right]^2 - \tfrac{3C}{R}\left[\left(\eta F\right)\right] \\
& + \left[n - \tfrac{bC}{R} - \left(\tfrac{2b}{na^2} + \tfrac{bC}{R}\right)\left(\eta F\right)\right]t^2\eta^2 f_0^2 \\
& + \left[-\tfrac{2(1+a)}{na^2}\left(\eta F\right) - 2k - \tfrac{4}{b} - \tfrac{C}{R}\right]t\eta\left|\nabla_b f\right|^2 \\
& + 4t_0^2\eta^2\left(\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2\right).
\end{aligned} \tag{3.16}
$$

We claim at $t=1$

$$(\eta F)\left(p\left(1\right),\ 1,\ R,\ b\right)<\frac{na^{2}}{-2\left(1+a\right)}\left(2k+\frac{4}{b}+\frac{C}{R}\right) \tag{3.17}$$

for a large enough R which to be determined later. Here $(1+a)<0$ for some a to be chosen later (say $1+a=-\frac{5+2bk}{n}$).

We prove it by contradiction. Suppose not, that is

$$(\eta F)\left(p\left(1\right),\ 1,\ R,\ b\right)\geq\frac{na^{2}}{-2\left(1+a\right)}\left(2k+\frac{4}{b}+\frac{C}{R}\right).$$

Since $(\eta F)\left(p\left(t\right),\ t,\ R,\ b\right)$ is continuous in the variable t and $(\eta F)\left(p\left(0\right),\ 0,\ R,\ b\right)=0$, by Intermediate-value theorem there exists a $t_0\in(0,\ 1]$ such that

$$(\eta F)\left(p\left(t_0\right),\ t_0,\ R,\ b\right)=\frac{na^{2}}{-2\left(1+a\right)}\left(2k+\frac{4}{b}+\frac{C}{R}\right). \tag{3.18}$$

Now we apply (3.16) at the point $\left(p\left(t_0\right),t_0\right)$, denoted by $\left(p_0,t_0\right)$. We have by using (3.18)

$$\begin{array}{rcl} 0 & \geq & \left(\frac{1}{na^2}-\frac{C}{R}\right)\left[(\eta F)\left(p_0,t_0\right)\right]^2-\frac{3C}{R}\left[(\eta F)\left(p_0,t_0\right)\right] \\ & & +\left[n-\frac{bC}{R}-\left(\frac{2b}{na^2}+\frac{bC}{R}\right)(\eta F)\left(p_0,t_0\right)\right]t^2\eta^2 f_0^2 \\ & & +4t_0^2\eta^2\left(\sum\limits_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2+\sum\limits_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2\right). \end{array} \tag{3.19}$$

Moreover, we compute

$$\begin{array}{l} \left[\left(\frac{1}{na^2}-\frac{C}{R}\right)(\eta F)\left(p_0,t_0\right)-\frac{3C}{R}\right] \\ =\left[\left(\frac{1}{na^2}-\frac{C}{R}\right)\left(\frac{na^2}{-2(1+a)}\right)\left(2k+\frac{4}{b}+\frac{C}{R}\right)-\frac{3C}{R}\right] \\ =\left\{\frac{-1}{2(1+a)}\left(2k+\frac{4}{b}\right)-\frac{C}{R}\left[\frac{na^2}{-2(1+a)}\left(2k+\frac{4}{b}+\frac{C}{R}\right)+\frac{1}{2(1+a)}+3\right]\right\} \end{array} \tag{3.20}$$

and

$$\begin{array}{l} \left[n-\frac{bC}{R}-\left(\frac{2b}{na^2}+\frac{bC}{R}\right)(\eta F)\left(p_0,t_0\right)\right] \\ =n-\frac{bC}{R}-\left(\frac{2b}{na^2}+\frac{bC}{R}\right)\left(\frac{na^2}{-2(1+a)}\right)\left(2k+\frac{4}{b}+\frac{C}{R}\right) \\ =n-\frac{bC}{R}+\frac{b}{(1+a)}\left(2k+\frac{4}{b}+\frac{C}{R}\right)+\frac{bC}{R}\left(\frac{na^2}{2(1+a)}\right)\left(2k+\frac{4}{b}+\frac{C}{R}\right) \\ =\left(n+\frac{4}{1+a}+\frac{2bk}{1+a}\right)+\frac{C}{R}\left[-\frac{ab}{1+a}+\frac{na^2b}{2(1+a)}\left(2k+\frac{4}{b}+\frac{C}{R}\right)\right]. \end{array} \tag{3.21}$$

Now we choose a such that

$$(1+a) < -\frac{4+2bk}{n}$$

and then

$$\left(n + \frac{4}{1+a} + \frac{2bk}{1+a}\right) > 0.$$

In particular, we let

$$1+a = -\frac{5+2bk}{n}. \tag{3.22}$$

Then for $R = R(b,k)$ large enough, one obtains

$$\left[\left(\frac{1}{na^2} - \frac{C}{R}\right)(\eta F)(p_0,t_0) - \frac{3C}{R}\right] > 0$$

and

$$\left[n - \frac{bC}{R} - \left(\frac{2b}{na^2} + \frac{bC}{R}\right)(\eta F)(p_0,t_0)\right] > 0.$$

This leads to a contradiction with (3.19). Hence from (3.17) and (3.22)

$$(\eta F)(1,\ p(1),\ R,\ b) < \frac{(n+5+2bk)^2}{2(5+2bk)}\left(2k + \frac{4}{b} + \frac{C}{R}\right).$$

This implies

$$\max_{x\in B(2R)}\left(|\nabla_b f|^2 + b\eta f_0^2\right)(x) < \frac{(n+5+2bk)^2}{2(5+2bk)}\left(2k + \frac{4}{b} + \frac{C}{R}\right).$$

When we fix on the set $x \in B(R)$, we obtain

$$|\nabla_b f|^2 + bf_0^2 < \frac{(n+5+2bk)^2}{2(5+2bk)}\left(2k + \frac{4}{b} + \frac{C}{R}\right)$$

on $B(R)$.

This completes the proof.

Next we prove Theorem 8. The proof is similar to Theorem 4.

Proof of Theorem 8 :

Proof. Firstly, we recall (Proposition 15) that

$$
\begin{aligned}
& t\eta\Delta_b(\eta F) \\
\geq\ & \tfrac{1}{na^2}(\eta F)^2 - \tfrac{C}{R}(\eta F) + 2t\eta\langle\nabla_b\eta,\ \nabla_b F\rangle - 2t\eta^2\langle\nabla_b f,\ \nabla_b F\rangle \\
& +4t^2\eta^2\left(\sum_{\alpha,\beta=1}^{n}|f_{a\beta}|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2\right) \\
& +\left[n - \tfrac{bC}{R} - \left(\tfrac{2b}{na^2} + \tfrac{bC}{R}\right)(\eta F)\right]t^2\eta^2 f_0^2 \\
& +\left[-\tfrac{2(1+a)}{na^2}(\eta F) - 2k - \tfrac{4}{b}\right]t\eta|\nabla_b f|^2 + 4bt^3\eta^3 f_0 V(f).
\end{aligned}
\tag{3.23}
$$

Now we need to deal with the term $4bt^3\eta^3 f_0 V(f)$ in (3.23).

(3.24)

$$
\begin{aligned}
& 4bt^3\eta^3 f_0 V(f) \\
&= 4bt^3\eta^3 f_0\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar{\beta}}\right)_{\bar{\alpha}} + \left(A_{\bar{\alpha}\bar{\beta}}f_{\beta}\right)_{\alpha} + A_{\alpha\beta}f_{\bar{\alpha}}f_{\bar{\beta}} + A_{\bar{\alpha}\bar{\beta}}f_{\alpha}f_{\beta}\right] \\
&= 4bt^3\eta^3 f_0\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar{\beta}\bar{\alpha}} + A_{\bar{\alpha}\bar{\beta}}f_{\beta\alpha}\right) + \left(A_{\alpha\beta,\bar{\alpha}}f_{\bar{\beta}} + A_{\bar{\alpha}\bar{\beta},\alpha}f_{\beta}\right) + \left(A_{\alpha\beta}f_{\bar{\alpha}}f_{\bar{\beta}} + A_{\bar{\alpha}\bar{\beta}}f_{\alpha}f_{\beta}\right)\right] \\
&\geq -8bt^3\eta^3|f_0|\sum_{\alpha,\beta=1}^{n}\left(\left|A_{\bar{\alpha}\bar{\beta}}\right||f_{\beta\alpha}| + |A_{\alpha\beta,\bar{\alpha}}|\left|f_{\bar{\beta}}\right| + \left|A_{\bar{\alpha}\bar{\beta}}\right||f_{\alpha}||f_{\beta}|\right)
\end{aligned}
$$

In (3.24), by Young's inequality and noting that $t \leq 1$, $\eta \leq 1$, we have following estimates:

$$
\begin{aligned}
-8bt^3\eta^3|f_0|\sum_{\alpha,\beta=1}^{n}\left|A_{\bar{\alpha}\bar{\beta}}\right||f_{\beta\alpha}| &\geq \sum_{\alpha,\beta=1}^{n} - 8k_1bt^3\eta^3|f_0||f_{\beta\alpha}| \\
&\geq \sum_{\alpha,\beta=1}^{n}\left(-4k_1bt^3\eta^3|f_{\beta\alpha}|^2 - 4k_1bt^3\eta^3 f_0^2\right) \\
&\geq -4k_1bn^2\left(t^2\eta^2 f_0^2\right) - 4k_1bt^2\eta^2\sum_{\alpha,\beta=1}^{n}|f_{\beta\alpha}|^2
\end{aligned}
\tag{3.25}
$$

and

$$
\begin{aligned}
-8bt^3\eta^3|f_0|\sum_{\alpha,\beta=1}^{n}|A_{\alpha\beta,\bar{\alpha}}|\left|f_{\bar{\beta}}\right| &\geq -8k_1bt^3\eta^3\sum_{\alpha,\beta=1}^{n}|f_0|\left|f_{\bar{\beta}}\right| \\
&\geq -8k_1bt^3\eta^3\sum_{\alpha,\beta=1}^{n}\left(\tfrac{1}{2}f_0^2 + \tfrac{1}{2}\left|f_{\bar{\beta}}\right|^2\right) \\
&\geq -4k_1bn^2t^3\eta^3 f_0^2 - 4k_1bnt^3\eta^3\sum_{\beta=1}^{n}\left|f_{\bar{\beta}}\right|^2 \\
&\geq -4k_1bn^2\left(t^2\eta^2 f_0^2\right) - 2k_1bn\left(t\eta|\nabla_b f|^2\right)
\end{aligned}
\tag{3.26}
$$

and

(3.27)

$$\begin{aligned}
-8bt^3\eta^3\left|f_0\right|\sum_{\alpha,\beta=1}^{n}\left|A_{\bar{\alpha}\bar{\beta}}\right|\left|f_\alpha\right|\left|f_\beta\right| &\geq -8k_1bt^3\eta^3\left|f_0\right|\sum_{\alpha,\beta=1}^{n}\left(\tfrac{1}{2}\left|f_\alpha\right|^2+\tfrac{1}{2}\left|f_\beta\right|^2\right)\\
&\geq -4k_1bt^3\eta^3\left|f_0\right|\left(n\sum_{\alpha=1}^{n}\left|f_\alpha\right|^2+n\sum_{\beta=1}^{n}\left|f_\beta\right|^2\right)\\
&\geq -4k_1bnt^3\eta^3\left|f_0\right|\left|\nabla_bf\right|^2\\
&\geq -2k_1b^2nt^3\eta^3f_0^2\left|\nabla_bf\right|^2-2k_1nt^3\eta^3\left|\nabla_bf\right|^2\\
&= -2k_1b^2n\left(t\eta\left|\nabla_bf\right|^2\right)\left(t^2\eta^2f_0^2\right)-2k_1nt^3\eta^3\left|\nabla_bf\right|^2.
\end{aligned}$$

Substitute estimates (3.25), (3.26), and (3.27) into (3.23), one obtains

$$\begin{aligned}
t\eta\Delta_b\left(\eta F\right) \geq\ & \tfrac{1}{na^2}\left(\eta F\right)^2-\tfrac{C}{R}\left(\eta F\right)+2t\eta\left\langle\nabla_b\eta,\ \nabla_bF\right\rangle-2t\eta^2\left\langle\nabla_bf,\ \nabla_bF\right\rangle\\
&+4t^2\eta^2\left[\left(1-bk_1n\right)\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2+\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2\right]\\
&+\left[n-8bk_1n^2-\tfrac{bC}{R}-\left(\tfrac{2b}{na^2}+2b^2k_1n+\tfrac{bC}{R}\right)\left(\eta F\right)\right]t^2\eta^2f_0^2\\
&+\left[-\tfrac{2(1+a)}{na^2}\left(\eta F\right)-2k-2n\left(1+b\right)k_1-\tfrac{4}{b}\right]t\eta\left|\nabla_bf\right|^2.
\end{aligned}$$

Next as shown in the same computation as in Proposition 16, at the maximal point $(p(t),\ t)$

$$\begin{aligned}
0 \geq\ & \left(\tfrac{1}{na^2}-\tfrac{C}{R}\right)\left(\eta F\right)^2-\tfrac{3C}{R}\left(\eta F\right)\\
&+4t^2\eta^2\left[\left(1-bk_1n\right)\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2+\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2\right]\\
&+\left[n-8bk_1n^2-\tfrac{bC}{R}-\left(\tfrac{2b}{na^2}+2b^2k_1n+\tfrac{bC}{R}\right)\left(\eta F\right)\right]t^2\eta^2f_0^2\\
&+\left[-\tfrac{2(1+a)}{na^2}\left(\eta F\right)-2k-2n\left(1+b\right)k_1-\tfrac{4}{b}-\tfrac{C}{R}\right]t\eta\left|\nabla_bf\right|^2.
\end{aligned}\tag{3.28}$$

We claim at $t=1$, there exists a small constant $b_0=b_0(n,k,k_1)>0$ such that for any $0<b\leq b_0$

$$\left(\eta F\right)\left(p\left(1\right),\ 1,\ R,\ b\right)<\frac{na^2}{-2\left(1+a\right)}\left(2k+2n\left(1+b\right)k_1+\frac{4}{b}+\frac{C}{R}\right)$$

if R is large enough which to be determined later. Here $(1+a)<0$ for some a to be chosen later (say $1+a=-\frac{5}{n}$).

We prove it by contradiction. Suppose not, that is

$$(\eta F)\left(p\left(1\right),\ 1,\ R,\ b\right)\geq\frac{na^{2}}{-2\left(1+a\right)}\left(2k+2n\left(1+b\right)k_{1}+\frac{4}{b}+\frac{C}{R}\right).$$

Since $(\eta F)\left(p\left(t\right),\ t,\ R,\ b\right)$ is continuous in the variable t and $(\eta F)\left(p\left(0\right),\ 0,\ R,\ b\right)=0$, by Intermediate-value theorem there exists a $t_0\in(0,\ 1]$ such that

$$(\eta F)\left(p\left(t_{0}\right),\ t_{0},\ R,\ b\right)=\frac{na^{2}}{-2\left(1+a\right)}\left(2k+2n\left(1+b\right)k_{1}+\frac{4}{b}+\frac{C}{R}\right).$$

Now we apply (3.28) at the point $\left(p\left(t_{0}\right),t_{0}\right)$, denoted by (p_0,t_0). We have

(3.29)
$$\begin{aligned}
&\left[\left(\tfrac{1}{na^{2}}-\tfrac{C}{R}\right)(\eta F)(p_{0},t_{0})-\tfrac{3C}{R}\right]\\
&=\left[\left(\tfrac{1}{na^{2}}-\tfrac{C}{R}\right)\left(\tfrac{na^{2}}{-2(1+a)}\right)\left(2k+2n\left(1+b\right)k_{1}+\tfrac{4}{b}+\tfrac{C}{R}\right)-\tfrac{3C}{R}\right]\\
&=\left\{\tfrac{-1}{2(1+a)}\left(2k+2n\left(1+b\right)k_{1}+\tfrac{4}{b}\right)-\tfrac{C}{R}\left[\tfrac{na^{2}}{-2(1+a)}\left(2k+2n\left(1+b\right)k_{1}+\tfrac{4}{b}+\tfrac{C}{R}\right)+\tfrac{1}{2(1+a)}+3\right]\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\left[n-8bk_{1}n^{2}-\tfrac{bC}{R}-\left(\tfrac{2b}{na^{2}}+2b^{2}k_{1}n+\tfrac{bC}{R}\right)(\eta F)(p_{0},t_{0})\right]\\
&=n-8bk_{1}n^{2}-\tfrac{bC}{R}-\left(\tfrac{2b}{na^{2}}+2b^{2}k_{1}n+\tfrac{bC}{R}\right)\left(\tfrac{na^{2}}{-2(1+a)}\right)\left(2k+2n\left(1+b\right)k_{1}+\tfrac{4}{b}+\tfrac{C}{R}\right)\\
&=n-8bk_{1}n^{2}-\tfrac{bC}{R}+\left(\tfrac{na^{2}}{2(1+a)}\right)\left(\tfrac{2b}{na^{2}}+2b^{2}k_{1}n\right)\left(2k+2n\left(1+b\right)k_{1}+\tfrac{4}{b}+\tfrac{C}{R}\right)\\
&+\tfrac{bC}{R}\left(\tfrac{na^{2}}{2(1+a)}\right)\left(2k+2n\left(1+b\right)k_{1}+\tfrac{4}{b}+\tfrac{C}{R}\right)\\
&=\left\{n-8bk_{1}n^{2}+\left(\tfrac{b+a^{2}b^{2}n^{2}k_{1}}{(1+a)}\right)\left[2k+2n\left(1+b\right)k_{1}+\tfrac{4}{b}\right]\right\}\\
&+\tfrac{C}{R}\left\{-b+\left(\tfrac{b+a^{2}b^{2}n^{2}k_{1}}{(1+a)}\right)+\tfrac{na^{2}b}{2(1+a)}\left[2k+2n\left(1+b\right)k_{1}+\tfrac{4}{b}+\tfrac{C}{R}\right]\right\}.
\end{aligned}\tag{3.30}$$

Now we choose a and b such that

$$\begin{aligned}
&n-8bk_{1}n^{2}+\left(\tfrac{b+a^{2}b^{2}n^{2}k_{1}}{(1+a)}\right)\left[2k+2n\left(1+b\right)k_{1}+\tfrac{4}{b}\right]\\
&=n-b\left\{8k_{1}n^{2}-\left(\tfrac{1+a^{2}bn^{2}k_{1}}{(1+a)}\right)\left[2k+2n\left(1+b\right)k_{1}\right]-\left(\tfrac{4a^{2}n^{2}k_{1}}{1+a}\right)\right\}+\tfrac{4}{1+a}\\
&>0.
\end{aligned}\tag{3.31}$$

This can be done by choosing

$$(1+a)<-\frac{4}{n}$$

and then choose a small $b_0 = b_0(n, k, k_1) > 0$ such that for any $b \leq b_0$

$$n - b\{8k_1n^2 - (\frac{1 + a^2bn^2k_1}{(1 + a)})[2k + 2n\,(1 + b)\,k_1] - (\frac{4a^2n^2k_1}{1 + a})\} + \frac{4}{1 + a} > 0$$

and

$$(1 - bk_1n) > 0.$$

In particular, we let

$$1 + a = -\frac{5}{n}.$$

Then for any $0 < b \leq b_0$, one obtains

$$\left[\left(\frac{1}{na^2} - \frac{C}{R}\right)(\eta F)\,(p_0, t_0) - \frac{3C}{R}\right] > 0$$

and

$$\left[n - 8bk_1n^2 - \frac{bC}{R} - \left(\frac{2b}{na^2} + 2b^2k_1n + \frac{bC}{R}\right)(\eta F)\,(p_0, t_0)\right] > 0$$

for $R = R(b, k, k_1)$ large enough. This leads to a contradiction with (3.28). Hence

$$(\eta F)\,(1,\ p\,(1)\,,\ R,\ b) < \frac{na^2}{-\,(1 + a)}\left(k + n\,(1 + b)\,k_1 + \frac{2}{b} + \frac{C}{R}\right).$$

This implies for $1 + a = -\frac{5}{n}$

$$\max_{x \in B(2R)}\left(|\nabla_b f|^2 + b\eta f_0^2\right)(x) < \frac{(n + 5)^2}{5}\left(k + n\,(1 + b)\,k_1 + \frac{2}{b} + \frac{C}{R}\right).$$

When we fix on the set $x \in B\,(R)$, we obtain

$$|\nabla_b f|^2 + bf_0^2 < \frac{(n + 5)^2}{5}\left(k + n\,(1 + b)\,k_1 + \frac{2}{b} + \frac{C}{R}\right)$$

on $B\,(R)$. Note that the preceding computation is not valid if ηF is not smooth at x_0. In this case, we may use a trick due to E. Calabi (see [W] for details).

This completes the proof of Theorem 8.

4. The Sub-Laplacian Comparison Theorem

In this section, we give the proof of sub-Laplacian comparison theorems in a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing pseudohermitian torsion tensors and nonnegative pseudohermitian Ricci curvature tensors. In order to prove Theorem 3, we first derive the differential inequality for sub-Laplacian of Carnot-Caratheodory distance.

For simplicity, we prove the propositions for $n=1$. In the setting, we write $Z_1 = \frac{1}{2}(e_1 - ie_2)$ for real vectors e_1, e_2. It follows $e_2 = Je_1$. Let $e^1 = \mathrm{Re}(\theta^1)$, $e^2 = \mathrm{Im}(\theta^1)$. Then $\{\theta, e^1, e^2\}$ is dual to $\{T, e_1, e_2\}$. Now in view of (2.1) and (2.3), we have the following real version of structure equations:

$$
\begin{aligned}
d\theta &= 2e^1 \wedge e^2, \\
\nabla e_1 &= \omega \otimes e_2,\ \nabla e_2 = -\omega \otimes e_1, \\
de^1 &= -e^2 \wedge \omega \bmod \theta;\ de^2 = e^1 \wedge \omega \bmod \theta.
\end{aligned}
$$

We also write $\varphi_{e_i} = e_i\varphi$ and $\nabla_b\varphi = \frac{1}{2}(\varphi_{e_1}e_1 + \varphi_{e_2}e_2)$. Moreover, $\varphi_{e_ie_j} = e_je_i\varphi - \nabla_{e_j}e_i\varphi$ and $\Delta_b\varphi = \frac{1}{2}(\varphi_{e_1e_1} + \varphi_{e_2e_2})$. Now we can write down the real version of the commutation relations as in 2.5 and 2.6.

$$
\begin{aligned}
\varphi_{e_1e_2} - \varphi_{e_2e_1} &= 2\varphi_0 \\
\varphi_{0e_1} - \varphi_{e_10} &= \varphi_{e_1}\,\mathrm{Re}\,A_{11} - \varphi_{e_2}\,\mathrm{Im}\,A_{11} \\
\varphi_{0e_2} - \varphi_{e_20} &= \varphi_{e_1}\,\mathrm{Im}\,A_{11} + \varphi_{e_2}\,\mathrm{Re}\,A_{11} \\
\varphi_{e_1e_1e_2} - \varphi_{e_1e_2e_1} &= 2\varphi_{e_10} - 2\varphi_{e_2}W \\
\varphi_{e_2e_1e_2} - \varphi_{e_2e_2e_1} &= 2\varphi_{e_20} + 2\varphi_{e_1}W.
\end{aligned}
\tag{4.1}
$$

For a fixed point $p \in M$, we consider the Carnot-Carathéodory distance function $r_p(x) = r(p,x)$ from x to p, and we will simply write $r(x)$. The distance function in general is not smooth due to the presence of cut-points. However, it can be seen that it is a Lipschitz

function with Lipschitz constant 1. In particular, we have

$$|\nabla r|^2 = 1$$

almost everywhere on M. Though r might not be a C^2-function, one can still estimate its sub-Laplacian in the sense of distribution ([BGG], [S]).

Definition 4. Let (M, J, θ) be a pseudohermitian 3-manifold. We define ([L1])

$$P\varphi = (\varphi_{\bar{1}11} + iA_{11}\varphi^1)\theta^1 = P\varphi = (P_1\varphi)\theta^1,$$

which is an operator that characterizes CR-pluriharmonic functions. Here $P_1\varphi = \varphi_{\bar{1}11} + iA_{11}\varphi^1$ and $\overline{P}\varphi = (\overline{P}_1)\theta^{\bar{1}}$, the conjugate of P.

Lemma 17. Let (M, J, θ) be a complete pseudohermitian 3-manifold. Then

$$\partial_r(\Delta_b r) + 2(\Delta_b r)^2 - 2r_{0e_1} + 2r_0^2 + (W - \operatorname{Im} A_{11}) = 0 \tag{4.2}$$

and

$$r_{00e_2} + r_{0e_1}^2 - (\operatorname{Re} A_{11})^2 - (\operatorname{Re} A_{11})_0 = 0. \tag{4.3}$$

As a consequence, we have

$$3\partial_r(\Delta_b r) + 2(\Delta_b r)^2 + 2r_0^2 - 8 < \Pr + \overline{P}r, d_b r > +(W + 3\operatorname{Im} A_{11}) = 0. \tag{4.4}$$

Here $d_b r = r_1\theta^1 + r_{\bar{1}}\theta^{\bar{1}}$.

Proof. We will follow the method as in ([W]). Let $x \in \exp_p U_p \backslash \{p\}$. Here $\exp_p$ is the exponential map due to R. Strichartz ([S]). Let γ be the minimal geodesic joining p to x. As in [S], the CR Gauss lemma implies that one can choose a CR orthonormal frame $\{T, e_1, e_2\}$ along γ such that $\nabla r = e_2$. Then

$$|\nabla r| = r_{e_2} = \partial_r r = 1 \tag{4.5}$$

and

$$r_{e_1} = 0 \quad \text{and} \quad r_{e_2e_1} = 0 = r_{e_2e_2}. \tag{4.6}$$

Hence

$$\begin{aligned} 0 &= (r_{e_1}^2 + r_{e_2}^2)_{e_1e_1} + (r_{e_1}^2 + r_{e_2}^2)_{e_2e_2} \\ &= 2(r_{e_1}r_{e_1e_1} + r_{e_2}r_{e_2e_1})_{e_1} + 2(r_{e_1}r_{e_1e_2} + r_{e_2}r_{e_2e_2})_{e_2} \\ &= 2(r_{e_1e_1}^2 + r_{e_1}r_{e_1e_1e_1} + r_{e_2e_1}^2 + r_{e_2}r_{e_2e_1e_1}) \\ &\quad +2(r_{e_1e_2}^2 + r_{e_1}r_{e_1e_2e_2} + r_{e_2e_2}^2 + r_{e_2}r_{e_2e_2e_2}) \\ &= 2(r_{e_1e_1}^2 + r_{e_2e_2}^2 + r_{e_2e_1}^2 + r_{e_1e_2}^2 + r_{e_2e_1e_1} + r_{e_2e_2e_2}). \end{aligned} \tag{4.7}$$

On the other hand from (4.1)

$$r_{e_1e_2}^2 = 4r_0^2\,, \quad r_{e_1e_1}^2 = 4(\Delta_b r)^2.$$

Moreover

$$\begin{aligned} r_{e_2e_1e_1} &= r_{e_1e_2e_1} - 2r_{0e_1} \\ &= r_{e_1e_1e_2} - 2r_{e_10} + 2r_{e_2}W - 2r_{0e_1} \\ &= r_{e_1e_1e_2} - 4r_{e_10} + 2r_{e_2}W - 2r_{e_1}\operatorname{Re}A_{11} + 2r_{e_2}\operatorname{Im}A_{11} \\ &= r_{e_1e_1e_2} - 4r_{e_10} + 2W + 2\operatorname{Im}A_{11}. \end{aligned}$$

All these and (4.7) imply

$$\begin{aligned} 0 &= r_{e_1e_1}^2 + r_{e_1e_2}^2 + r_{e_2e_1e_1} + r_{e_2e_2e_2} \\ &= 4(\Delta_b r)^2 + 4r_0^2 + (r_{e_1e_1} + r_{e_2e_2})_{e_2} - 4r_{e_10} + 2W + 2\operatorname{Im}A_{11} \\ &= 4(\Delta_b r)^2 + 4r_0^2 + 2\partial_r(\Delta_b r) - 4r_{e_10} + 2W + 2\operatorname{Im}A_{11}. \end{aligned}$$

That is

$$0 = \partial_r(\Delta_b r) + 2(\Delta_b r)^2 - 2r_{e_10} + 2r_0^2 + (W + \operatorname{Im}A_{11}). \tag{4.8}$$

But

$$r_{0e_1} - r_{e_10} = -\operatorname{Im}A_{11}. \tag{4.9}$$

Hence

$$0 = \partial_r(\Delta_b r) + 2(\Delta_b r)^2 - 2r_{0e_1} + 2r_0^2 + (W - \operatorname{Im} A_{11}).$$

On the other hand, since

$$\begin{aligned} 0 &= (r_{e_1}^2 + r_{e_2}^2)_0 \\ &= 2(r_{e_1} r_{e_1 0} + r_{e_2} r_{e_2 0}), \end{aligned}$$

it follows from (4.5) and (4.6)

$$r_{e_2 0} = 0. \tag{4.10}$$

Now as in (4.7), we have

$$\begin{aligned} 0 &= (r_{e_1}^2 + r_{e_2}^2)_{00} \\ &= 2(r_{e_1} r_{e_1 0} + r_{e_2} r_{e_2 0})_0 \\ &= 2(r_{e_1 0}^2 + r_{e_1} r_{e_1 00} + r_{e_2 0}^2 + r_{e_2} r_{e_2 00}) \\ &= 2(r_{e_1 0}^2 + r_{e_2 00}). \end{aligned}$$

Compute

$$r_{e_1 0}^2 = (r_{0e_1} + \operatorname{Im} A_{11})^2$$

and

$$\begin{aligned} r_{e_2 00} &= (r_{0e_2} - r_{e_1} \operatorname{Im} A_{11} - r_{e_2} \operatorname{Re} A_{11})_0 \\ &= r_{0e_2 0} - r_{e_1 0} \operatorname{Im} A_{11} - r_{e_1} (\operatorname{Im} A_{11})_0 - r_{e_2 0} \operatorname{Re} A_{11} - r_{e_2} (\operatorname{Re} A_{11})_0 \\ &= r_{0e_2 0} - r_{e_1 0} \operatorname{Im} A_{11} - (\operatorname{Re} A_{11})_0 \\ &= (r_{00e_2} - r_{0e_1} \operatorname{Im} A_{11} - r_{0e_2} \operatorname{Re} A_{11}) - (r_{0e_1} + \operatorname{Im} A_{11}) \operatorname{Im} A_{11} - (\operatorname{Re} A_{11})_0 \\ &= r_{00e_2} - 2r_{0e_1} \operatorname{Im} A_{11} - r_{0e_2} \operatorname{Re} A_{11} - (\operatorname{Im} A_{11})^2 - (\operatorname{Re} A_{11})_0. \end{aligned}$$

All these imply

$$0 = r_{00e_2} + r_{0e_1}^2 - r_{0e_2} \operatorname{Re} A_{11} - (\operatorname{Re} A_{11})_0 = r_{00e_2} + r_{0e_1}^2 - (\operatorname{Re} A_{11})^2 - (\operatorname{Re} A_{11})_0.$$

Finally, by definition we compute

$$< \operatorname{Pr} + \overline{P} r, d_b r >= (r_{\bar{1}11} r_{\bar{1}} + i A_{11} r_{\bar{1}} r_{\bar{1}}) + \text{conjugate}.$$

First we note that for $Z_1 = \frac{1}{2}(e_1 - ie_2)$

$$r_1 = \frac{1}{2}(r_{e_1} - ir_{e_2}) = -\frac{1}{2}i$$

and then

$$iA_{11}r_{\overline{1}}r_{\overline{1}} + \text{conjugate} = -\frac{1}{4}iA_{11} + \text{conjugate} = \frac{1}{2}\operatorname{Im} A_{11}. \tag{4.11}$$

Secondly, one can derive

$$\begin{aligned} & r_{\overline{1}11} \\ & = \tfrac{1}{8}[r_{e_1e_1e_1} + r_{e_2e_2e_1} + r_{e_2e_1e_2} - r_{e_1e_2e_2}] \\ & \quad + \tfrac{1}{8}i[(r_{e_2e_1e_1} - r_{e_1e_2e_1}) - (r_{e_1e_1e_2} + r_{e_2e_2e_2})] \end{aligned}$$

and

$$\begin{aligned} & r_{\overline{1}11}r_{\overline{1}} \\ & = \tfrac{1}{16}i[r_{e_1e_1e_1} + r_{e_2e_2e_1} + r_{e_2e_1e_2} - r_{e_1e_2e_2}] \\ & \quad - \tfrac{1}{16}[(r_{e_2e_1e_1} - r_{e_1e_2e_1}) - (r_{e_1e_1e_2} + r_{e_2e_2e_2})]. \end{aligned}$$

Hence

$$\begin{aligned} & r_{\overline{1}11}r_{\overline{1}} + \text{conjugate} \\ & = -\tfrac{1}{8}[(r_{e_2e_1e_1} - r_{e_1e_2e_1}) - (r_{e_1e_1e_2} + r_{e_2e_2e_2})] \\ & = \tfrac{1}{4}[\partial_r(\Delta_b r) + r_{0e_1}]. \end{aligned} \tag{4.12}$$

It follow from (4.11) and (4.12) that

$$4 < \Pr + \overline{P}r, d_b r >= \partial_r(\Delta_b r) + r_{0e_1} + 2\operatorname{Im} A_{11}$$

and then

$$3\partial_r(\Delta_b r) + 2(\Delta_b r)^2 + 2r_0^2 - 8 < \Pr + \overline{P}r, d_b r > +(W + 3\operatorname{Im} A_{11}) = 0.$$

□

Lemma 18. Let $(\mathbf{H}^n, J, \theta)$ be a standard Heisenberg $(2n+1)$-manifold. Then there exists a constant $C_1^{\mathbf{H}^n} > 0$

$$|r_{00}^{\mathbf{H}^n}| \leq C_{11}^{\mathbf{H}^n} (r^{\mathbf{H}^n})^{-3} \tag{4.13}$$

and

$$|r_{0}^{\mathbf{H}^n}| \leq C_{12}^{\mathbf{H}^n} (r^{\mathbf{H}^n})^{-1} \tag{4.14}$$

and

$$|r_{0e_2}^{\mathbf{H}^n}| \leq C_{13}^{\mathbf{H}^n} (r^{\mathbf{H}^n})^{-2} \tag{4.15}$$

We will give the proof of Lemma 18 at the end of this section.

Now **Theorem 3** will follows from the following Bishop-type sub-Laplacian Comparison Theorem easily.

Theorem 19. Let (M, J, θ) be a complete pseudohermitian 3-manifold of vanishing pseudohermitian torsion with

$$W \geq k_2$$

for some constant k_2. Then, for any $x \in M$ where $r(x)$ is smooth, we have

$$\Delta_b r \leq \begin{cases} m_2\sqrt{(1-\delta_1)k_2}\cot(\sqrt{(1-\delta_1)k_2}r), & k_2 > 0 \\ \frac{m_1}{r}, & k_2 = 0 \\ m_3\sqrt{(1+\delta_2)|k_2|}\coth(\sqrt{(1+\delta_2)|k_2|}r), & k_2 < 0 \end{cases} \tag{4.16}$$

for some positive constants $m_1,\ m_2,\ m_3$ and $\delta_1 < 1,\ \delta_2 < 1$. Moreover it holds on the whole manifold in the sense of distribution.

Remark 4. 1. Here we will apply the differential inequality for sub-Laplacian of Carnot-Caratheodory distance as in Lemma 17 to prove this sub-Laplacian comparison (4.16). In the first named author's previous paper ([CC]), we apply the same differential inequality to obtain the CR volume growth estimate.

2. In the paper of [AL], they obtained the weak Bishop-Laplacian comparison theorem in contact 3-manifolds.

Proof of Theorem 19 :

Proof. Here we will apply the differential inequality for sub-Laplacian of Carnot-Caratheodory distance as in Lemma 17. Since $A_{11} = 0$ and (4.3), we have

$$\partial_r r_{00} = -r_{0e_1}^2 \leq 0. \tag{4.17}$$

In particular

$$\partial_r (r_{00}) - \frac{C_{11}}{r^4} \leq 0. \tag{4.18}$$

Note that $r_{00} = r_{00}^{\mathbf{H}^1} + O(r)$ for a smaller r and $|r_{00}^{\mathbf{H}^1}| \leq C_{11}^{\mathbf{H}^1} (r^{\mathbf{H}^1})^{-3}$. By applying the method to the differential inequality (4.18) as in ([W]), it follows that

$$r_{00} \leq \frac{C'}{r^3} \tag{4.19}$$

for a larger r. By integrating both sides of (4.17) with respect to r, it follows from (4.19) that

$$|r_{0e_1}| \leq \frac{C''}{r^2}. \tag{4.20}$$

Hence

$$-\frac{l}{r^2} \leq -2r_{0e_1} + 2r_0^2$$

for some positive constant l.

(i) For $k_2 = 0$, it follows from (4.2) that

$$\partial_r(\Delta_b r) + 2(\Delta_b r)^2 - \frac{l}{r^2} \leq 0.$$

Then as in ([W])

$$\Delta_b r \leq \frac{m_1}{r}.$$

Here $m_1 = \frac{1+\sqrt{1+8l}}{4}$ solves the equation

$$2m^2 - m - l = 0.$$

(ii) For $k_2 > 0$, then from (4.4)

$$\partial_r(\Delta_b r) + 2(\Delta_b r)^2 - \frac{l}{r^2} + k_2 \leq 0.$$

Hence

$$\partial_r(\Delta_b r) + 2(\Delta_b r)^2 + (1-\delta_1)k_2 \leq 0 \tag{4.21}$$

if $r \geq \sqrt{\frac{l}{\delta_1 k_2}}$ with any positive constant $\delta_1 < 1$. Then by applying Wang's method again

$$\Delta_b r \leq m_2 \sqrt{K_2} \cot(\sqrt{K_2} r) \tag{4.22}$$

for some constant m_2 and $K_2 = (1-\delta_1)k_2$.

(iii) Similarly for $k_2 < 0$, we have

$$\partial_r(\Delta_b r) + 2(\Delta_b r)^2 - (1+\delta_2)|k_2| \leq 0$$

and if $r \geq \sqrt{\frac{l}{\delta_2 |k_2|}}$ with any positive constant $\delta_2 < 1$

$$\Delta_b r \leq m_3 \sqrt{K_3} \coth(\sqrt{K_3} r)$$

for some positive constant m_3 and $K_3 = (1+\delta_2)|k_2|$. □

Proof of Lemma 18 :

Proof. For simplicity, we prove (4.13) and (4.14) for $n = 1$. We consider the following two vector fields defined on $\mathbb{R}^3$ with coordinates $(\mathbf{x}, t) = (x_1, x_2, t)$:

$$X_1 = \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial t} \quad \text{and} \quad X_2 = \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial t}. \tag{4.23}$$

It is easy to check that

$$[X_1, X_2] = -4 \frac{\partial}{\partial t}. \tag{4.24}$$

The vector fields X_1, X_2 and $T = 2\frac{\partial}{\partial t}$ are left-invariant with respect to the "Heisenberg translation": for $(\mathbf{x}, t) = (x_1, x_2, t)$ and $(\mathbf{y}, s) = (y_1, y_2, s) \in \mathbb{R}^3$,

$$(\mathbf{x}, t) \circ (\mathbf{y}, s) = (x_1 + y_1, x_2 + y_2, t + s + 2[x_2 y_1 - x_1 y_2]). \tag{4.25}$$

Let $z = x_1 + ix_2$. Then we denote the vector fields in terms of complex coordinates as following :

$$Z = \frac{1}{2}(X_1 - iX_2) = \frac{\partial}{\partial z} + i\bar{z}\frac{\partial}{\partial t} \quad \text{and} \quad \bar{Z} = \frac{1}{2}(X_1 + iX_2) = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial t}.$$

Actually, the above multiplicative law defines a group structure on $\mathbb{R}^3$ which we call the 1-dimensional Heisenberg group with $(x, t)^{-1} = (-x, -t)$.

From the paper of [CTW] (also [BGG]), the square of the *Carnot-Caratheodory distance* $[r^{\mathbf{H}^1}(x, t)]^2$ from the origin is

$$[r^{\mathbf{H}^1}(x, t)]^2 = \left[\frac{2\theta_c}{\sin(2\theta_c)}\right]^2 \|x\|^2 = \nu(2\theta_c)\left(|t| + \|x\|^2\right), \tag{4.26}$$

where θ_c is the unique solution of $\mu(2\theta)\|x\|^2 = |t|$ in the interval $[0, \pi/2)$ and $\mu(z) = \frac{z}{\sin^2 z} - \cot z$. Moreover

$$\nu(z) = \frac{z^2}{\sin^2 z}\frac{1}{1 + \mu(z)} = \frac{z^2}{z + \sin^2 z - \sin z \cos z}; \ \nu(0) = 2.$$

.

Introduce a new parameter $\phi = 2\theta_c$. Then the Carnot-Cartheodory distance between the origin and point (x_1, x_2, t) can be expressed as

$$r^{\mathbf{H}^1}(x, t) = \frac{\phi}{\sin \phi}\|x\| \quad \text{with} \quad \mu(\phi)\|x\|^2 = |t| \quad \text{and} \quad \phi \in [0, \pi).$$

Denote

$$g(\phi) = \frac{\phi}{\sin \phi} \quad \text{and} \quad R = x_1^2 + x_2^2.$$

We have

$$\frac{\partial r^{\mathbf{H}^1}}{\partial t} = R^{1/2} g'(\phi) \frac{\partial \phi}{\partial t}$$

and

$$\frac{\partial^2 r^{\mathbf{H}^1}}{\partial t^2} = R^{1/2} g''(\phi) \left(\frac{\partial \phi}{\partial t}\right)^2 + R^{1/2} g'(\phi) \frac{\partial^2 \phi}{\partial t^2}.$$

Next from $t = R\mu(\phi)$, we compute the derivatives:

$$\frac{\partial \phi}{\partial t} = \frac{1}{R\mu'(\phi)} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial t^2} = -\frac{\mu''(\phi)}{R^2(\mu')^3}.$$

This implies

$$\frac{\partial r^{\mathbf{H}^1}}{\partial t} = R^{-1/2} \frac{g'(\phi)}{\mu'(\phi)}$$

and

$$\frac{\partial^2 r^{\mathbf{H}^1}}{\partial t^2} = R^{-3/2} \frac{1}{\mu'(\phi)} \left[\frac{g'(\phi)}{\mu'(\phi)}\right]'.$$

From the straightforward calculation, we have

$$\frac{1}{\mu'} \left(\frac{g'}{\mu'}\right)' = \frac{f}{2} \quad \text{with} \quad f = \frac{\sin^2 \phi \cos \phi}{2(1 - \phi \cot \phi)}.$$

This implies

$$\frac{\partial^2 r^{\mathbf{H}^1}}{\partial t^2} = \frac{f}{2} R^{-3/2}.$$

But $r^{\mathbf{H}^1}(R, t) = \frac{\phi}{\sin \phi} R^{1/2}$. We have

$$\frac{\partial^2 r^{\mathbf{H}^1}}{\partial t^2} = \frac{1}{4} R^{-3/2} \frac{\sin^2 \phi \cos \phi}{(1 - \phi \cot \phi)} = \frac{1}{4} (r^{\mathbf{H}^1})^{-3} \frac{\phi^3 \cos \phi}{\sin \phi (1 - \phi \cot \phi)}.$$

Hence

$$\left|\frac{\partial^2 r^{\mathbf{H}^1}}{\partial t^2}\right| \leq C (r^{\mathbf{H}^1})^{-3}.$$

Similarly, we have

$$\left|\frac{\partial r^{\mathbf{H}^1}}{\partial t}\right| \leq C (r^{\mathbf{H}^1})^{-1}.$$

At last, by integrating both sides of (4.17) with respect to r, (4.15) follows easily.

$\square$

Appendix A.

Here we derive the CR sub-Laplacian comparison property in a standard Heisenberg $(2n+1)$-manifold with $n = 1$. We refer to [CTW] for details with $n \geq 1$.

Denote $\Delta_b = \frac{1}{2}(X_1^2 + X_2^2)$. We will compute $\Delta_b r^{\mathbf{H}^1}(x, t)$ in polar coordinates as following.

$$\Delta_b = \frac{1}{2}(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial^2}{\partial s \partial \theta} + \frac{1}{s^2}\frac{\partial^2}{\partial \theta^2}) + 2\frac{\partial^2}{\partial t \partial \theta} + 2s^2\frac{\partial^2}{\partial t^2}.$$

Since $r^{\mathbf{H}^1}(x, t)$ depends only on $s = \|x\| = \sqrt{x_1^2 + x_2^2}$, we have

$$\Delta_b r^{\mathbf{H}^1}(x, t) = (\frac{1}{2}\frac{\partial^2}{\partial s^2} + 2s^2\frac{\partial^2}{\partial t^2})r^{\mathbf{H}^1}(s, t).$$

From (4.26), we introduce a new variable $u = |t|/s^2$. Then

$$r^{\mathbf{H}^1}(s, t) := f_c(s, u) = \frac{\phi}{\sin\phi}s \quad \text{where } u \text{ satisfies} \quad u = \mu(\phi) = \frac{\phi - \sin\phi\cos\phi}{\sin^2\phi}.$$

Hence

$$\Delta_b r^{\mathbf{H}^1}(s, t) = (\frac{1}{2}\frac{\partial^2}{\partial s^2} + \frac{2}{s^2}\frac{\partial^2}{\partial u^2})f_c(s, u) = \frac{2}{s}\frac{\partial^2}{\partial u^2}(\frac{\phi}{\sin\phi}),$$

where u is given by $u = \mu(\phi)$. Let $g(\phi) = \frac{\phi}{\sin\phi}$. Then

$$\frac{dg}{du} = \frac{dg}{d\phi}\cdot\frac{d\phi}{du} \quad \text{and} \quad \frac{d^2g}{du^2} = \frac{d^2g}{d\phi^2}\cdot(\frac{d\phi}{du})^2 + \frac{dg}{d\phi}\cdot\frac{d^2\phi}{du^2}.$$

We next compute $\frac{dg}{d\phi}$, $\frac{d^2g}{d\phi^2}$, $\frac{d\phi}{du}$ and $\frac{d^2\phi}{du^2}$.

$$\frac{dg}{d\phi} = \frac{\sin\phi - \phi\cos\phi}{\sin^2\phi} \quad \text{and} \quad \frac{d^2g}{d\phi^2} = \frac{\phi(1 + \cos^2\phi) - 2\sin\phi\cos\phi}{\sin^3\phi}.$$

Next $u = \mu(\phi)$ implies

$$1 = \mu'(\phi)\frac{d\phi}{du}, \quad \frac{d\phi}{du} = \frac{1}{\mu'(\phi)} \quad \text{and} \quad \frac{d^2\phi}{du^2} = -\frac{\mu''(\phi)}{(\mu')^3}.$$

We now compute $\mu'(\phi)$ and $\mu''(\phi)$ from $\mu(\phi) = \phi\csc^2\phi - \cot\phi$.

$$\mu'^2\phi - 2\phi\csc^2\phi\cot\phi + \csc^2\phi = 2\csc^2\phi(1 - \phi\cot\phi)$$

and

$$\mu''^2\phi\cot\phi(1-\phi\cot\phi)+2\csc^2\phi(\phi\csc^2\phi-\cot\phi)$$
$$=2\csc^2\phi[\phi(3\cot^2\phi+1)-3\cot\phi].$$

We finally compute $\Delta_b f_c(s,u)=\frac{2}{s}\frac{d^2}{du^2}g(\phi)$.

$$\begin{aligned}\Delta_b f_c(s,u) &= \frac{2}{s}\left[\frac{d^2g}{d\phi^2}\cdot(\frac{d\phi}{du})^2+\frac{dg}{d\phi}\cdot\frac{d^2\phi}{du^2}\right]\\ &= \frac{2}{s}\left[\frac{d^2g}{d\phi^2}\cdot\frac{1}{(\mu')^2}-\frac{dg}{d\phi}\frac{\mu''(\phi)}{(\mu')^3}\right]\\ &= \frac{2}{s(\mu')^2}\left[\frac{d^2g}{d\phi^2}-\frac{dg}{d\phi}\cdot\frac{\mu''(\phi)}{\mu'(\phi)}\right].\end{aligned}$$

We shall compute the term in $[\dots]$ in term of ϕ first.

$$\begin{aligned}&\frac{d^2g}{d\phi^2}-\frac{dg}{d\phi}\cdot\frac{\mu''(\phi)}{\mu'(\phi)}\\ =&\frac{\phi(1+\cos^2\phi)-2\sin\phi\cos\phi}{\sin^3\phi}-\frac{\sin\phi-\phi\cos\phi}{\sin^2\phi}\cdot\frac{2\csc^2\phi[\phi(3\cot^2\phi+1)-3\cot\phi]}{2\csc^2\phi(1-\phi\cot\phi)}\\ =&\frac{\phi(1+\cos^2\phi)-2\sin\phi\cos\phi}{\sin^3\phi}-\frac{\phi(3\cot^2\phi+1)-3\cot\phi}{\sin\phi}\\ =&\frac{\phi(1+\cos^2\phi)-2\sin\phi\cos\phi-\phi(3\cos^2\phi+\sin^2\phi)+3\cos\phi\sin\phi}{\sin^3\phi}\\ =&\frac{\sin\phi\cos\phi-\phi\cos^2\phi}{\sin^3\phi}.\end{aligned}$$

Hence we have

$$\begin{aligned}\Delta_b f_c(s,u) &= \frac{2}{s(\mu')^2}\left[\frac{d^2g}{d\phi^2}-\frac{dg}{d\phi}\cdot\frac{\mu''(\phi)}{\mu'(\phi)}\right]\\ &= \frac{\sin\phi\cos\phi-\phi\cos^2\phi}{\sin^3\phi}\cdot\frac{1}{2s\csc^4\phi(1-\phi\cot\phi)^2}\\ &= \frac{(1-\phi\cot\phi)\sin\phi\cos\phi}{2s\csc\phi(1-\phi\cot\phi)^2}\\ &= \frac{\sin^2\phi\cos\phi}{2s(1-\phi\cot\phi)}.\end{aligned}$$

Since $r^{\mathbf{H}^1} = \frac{\phi}{\sin\phi} s$,

$$\Delta_b r^{\mathbf{H}^1} = \frac{1}{2r^{\mathbf{H}^1}} \cdot \frac{\phi \sin^2\phi \cos\phi}{\sin\phi - \phi\cos\phi}. \tag{A.1}$$

We next study the function $F(\phi) = \frac{\phi\sin^2\phi\cos\phi}{2(\sin\phi - \phi\cos\phi)}$, where ϕ is given by

$$s^2\mu(\phi) = t \quad \text{with} \quad \mu(\phi) = \frac{\phi - \sin\phi\cos\phi}{\sin^2\phi}.$$

The function $F(\phi)$ is smooth on the interval $[0,\pi]$, decreasing from $[0,\phi_m]$ and increasing from $[\phi_m,\pi]$. ϕ_m is the unique critical point of $F(\phi)$ inside the interval $(0,\pi)$. $F(0) = 3$, $F(\pi/2) = F(\pi) = 0$.

As $s \to 0$ with $t > 0$ fixed, $\phi \to \pi^-$ and the equation $s^2\mu(\phi) = t$ implies

$$\frac{s^2}{t}\frac{\phi - \sin\phi\cos\phi}{\sin^2\phi} = 1.$$

This shows that

$$\phi \to \pi \quad \text{and} \quad \sin\phi \sim (\frac{\pi}{t})^{1/2} s \quad \text{as} \quad s \to 0.$$

This implies (A.1) makes sense when $s = 0$. This corresponds to $\phi = \pi$.

All these imply

$$\Delta_b r^{\mathbf{H}^1} \leq \frac{3}{r^{\mathbf{H}^1}}$$

in a standard Heisenberg 3-manifold.

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[1]Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan, [1]Taida Institute for Mathematical Sciences (TIMS), National Taiwan University, Taipei 10617, Taiwan

E-mail address: scchang@math.ntu.edu.tw

[2]Taida Institute for Mathematical Sciences (TIMS), National Taiwan University, Taipei 10617, Taiwan

E-mail address: tjkuo@ntu.edu.tw

[3]Department of Mathematics, University of Georgia, Athens, GA 30602-7403, U.S.A.

E-mail address: jtie@math.uga.edu